For the purpose of this article we define sovereign wealth funds (SWFs) as sovereign investment vehicles (returns enter the government's fiscal budget) with high foreign asset exposure, nonstandard liabilities, and a long (intergenerational) time horizon. In this article we focus on SWFs sourced by oil revenue as the currently most important (biggest) fraction of this class of new investors as can be seen in Exhibit 1.

Among the 10 biggest SWFs, we find eight funds that are sourced from oil revenue. Given an estimated market size of about $3 trillion at the beginning of 2008, the three biggest oil revenue funds account for 52% of total SWF assets. Given the long-term mediocre performance of spot oil (underground wealth), SWFs have been created to perform an oil to equity transformation to participate in global growth. The speed of this transformation will depend on the optimal path of extraction, which depends on the impact of increased supply on oil prices, extraction costs (technology), and oil price expectations. Given an estimated $40 trillion value of underground oil compared with $50 trillion in global equities, SWFs will have a major impact on global equity markets. They will also lead to a shift from traditional reserve currencies (dollar, yen) to emerging market currencies, where much of the global growth is to be expected.

For many oil exporting countries, crude oil or gas reserves are the single most important national asset. Any change in the value of reserves directly and materially affects these countries' wealth and thus the well-being of their citizens. Exhibit 2 serves as an illustration. Oil price changes are of violent nature and can have a destabilizing effect on the economy via volatile real exchange rates. Having recognized this, a number of oil exporting countries have been depositing oil revenues in funds dedicated to future expenditure. Devising optimal investment policies for such oil revenue funds is the aim of this article. We analyze optimal allocations among consists of financial assets and oil reserves. Finance) Summary, is “[f]irst, […] to act as a buffer to smooth short term variations in the oil revenues [in the Fiscal Budget, and second to] serve as a tool for coping with the financial challenges connected to an aging population and the eventual decline in oil revenues, by transferring wealth to future generations.” The second objective is operationalized as “[…] invest[ing] the capital in such a way that the fund’s international purchasing power is maximized, taking into account an acceptable level of risk.” This suggests that the benchmark of the fund is future consumption.
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Having recognized this, a number of oil exporting countries have been depositing oil revenues in funds dedicated to future expenditure. Devising optimal investment policies for such oil revenue funds is the aim of this article. We analyze optimal allocations among standard partitions of the investment universe, taking into account that aggregate wealth consists of financial assets and oil reserves.

An example of an oil revenue fund is Norway’s State Petroleum Fund. The policy goals of the fund, as stated in the Norske Finansdepartementet’s (Norwegian Ministry of Finance) Summary, is “[f]irst, […] to act as a buffer to smooth short term variations in the oil revenues [in the Fiscal Budget, … and second to] serve as a tool for coping with the financial challenges connected to an aging population and the eventual decline in oil revenues, by transferring wealth to future generations.” The second objective is operationalized as “[…] invest[ing] the capital in such a way that the fund’s international purchasing power is maximized, taking into account an acceptable level of risk.” This suggests that the benchmark of the fund is future consumption.
in the form of imports. The same reason also motivates
the inclusion of equity, which is expected to enhance
the performance of the fund. Concerning the definition of
risk, it appears that the Finansdepartementet is mostly
concerned with changes in the market value of the fund.
We were not able to infer the Finansdepartementet's views
on operationalizing the first objective, smoothing oil rev-
enues in the short term. We believe that both objectives,
smoothing revenues and maximizing long-term welfare,
suggest the more extensive definition of risk we propose
in this article.3

More generally, our article is an example of how
risk stemming from nonfinancial assets can be hedged,
at least partially, through financial assets. In other words,
we talk about asset allocation with nontradable wealth.
The key is exploiting the correlation between financial
and nonfinancial assets to reduce the overall risk of the
portfolio, compared to an allocation that considers only
the correlation structure of the financial assets. Although
the general idea is straightforward, empirical or practical
implementations are rare. An exception is asset/liability
management, in which interest rate exposure on one side
of the balance sheet is offset by interest rate exposure
on the other side. This article applies a similar idea to a
more general problem.

We will focus on portfolio investments. This is a
narrower brief than what sovereign investors can do.
Rather than investing into securities (mostly USD dom-
ninated) abroad, sovereign investors can also use their oil
revenues to build exposure to future growth industries
and develop the necessary infrastructure to make their
country an attractive place to attract top human talent.

Dubai and Qatar are prime examples of this.

During the following exposition we will rely on the
normality in return distributions assumption to allow us to
come up with closed-form solutions that provide concep-
tual insight into the structure of the underlying problem.
While we are aware that returns on capital markets and
certainly on commodities like oil are in the short run far
from normal, we also believe an SWF belongs to the group
of long-term investors such that the central limit theorem
will help somewhat to mitigate the nonnormality issue.

INCORPORATING THE SOVEREIGN WEALTH
FUND INTO GOVERNMENT BUDGETS

We view the optimal asset allocation problem of
an SWF as the decision-making problem of an investor

Note: The underlying total wealth position of an oil-rich country can vary
dramatically over time and needs management to smooth intergenerational
consumption patterns.

Note: All numbers are in billion dollars and based on public sources or our
own estimates as of the end of 2007.

<table>
<thead>
<tr>
<th>Sovereign</th>
<th>Assets</th>
<th>Inception</th>
<th>Source</th>
<th>Weight</th>
</tr>
</thead>
<tbody>
<tr>
<td>UAE</td>
<td>880</td>
<td>1976</td>
<td>Oil</td>
<td>29.33%</td>
</tr>
<tr>
<td>Norway</td>
<td>390</td>
<td>1996</td>
<td>Oil</td>
<td>13.00%</td>
</tr>
<tr>
<td>Singapore</td>
<td>350</td>
<td>1981</td>
<td>Misc</td>
<td>11.67%</td>
</tr>
<tr>
<td>Saudi Arabia</td>
<td>290</td>
<td>1981</td>
<td>Oil</td>
<td>9.67%</td>
</tr>
<tr>
<td>Kuwait</td>
<td>245</td>
<td>1953</td>
<td>Oil</td>
<td>8.17%</td>
</tr>
<tr>
<td>China</td>
<td>200</td>
<td>2007</td>
<td>Misc</td>
<td>6.67%</td>
</tr>
<tr>
<td>Libya</td>
<td>55</td>
<td>1974</td>
<td>Oil</td>
<td>1.83%</td>
</tr>
<tr>
<td>Qatar</td>
<td>49</td>
<td>N/A</td>
<td>Oil</td>
<td>1.63%</td>
</tr>
<tr>
<td>Algeria</td>
<td>44</td>
<td>2000</td>
<td>Oil</td>
<td>1.47%</td>
</tr>
<tr>
<td>U.S. (Alaska)</td>
<td>39</td>
<td>1976</td>
<td>Oil</td>
<td>1.30%</td>
</tr>
</tbody>
</table>

EXHIBIT 1
The 10 Biggest SWFs: Size and Source of Funding

EXHIBIT 2
Daily Oil Price Movements, January 1982
to September 2008
with nontradable endowed wealth (oil reserves). In order to get insight into the portfolio choice problem for an SWF, we assume the following analytical setup.

The SWF can invest its financial wealth into a single asset or cash. We can think of this as the choice between the global market portfolio and cash. This is certainly restrictive, but it will allow us to develop our framework without very complex calculations, and we will relax this assumption in the following section. Returns for this performance asset are normally distributed and given by

\[ \tilde{r}_s \sim N(\mu_s, \sigma_s^2) \]  

where \( \mu_s \) represents the expected risk premium (over local cash returns) of our performance asset and \( \sigma_s \) its volatility. At the same time, the government budget moves with changes on its claim on economic net wealth. For a commodity-based (oil-based) SWF, changes in the commodity (oil prices) will by far have the biggest influence on the government budget measured in economic (not accounting) terms. We assume that oil price changes are also normally distributed

\[ \tilde{r}_o \sim N(\mu_o, \sigma_o^2) \]  

and correlate positively with asset returns, \( \text{Cov}(\tilde{r}_s, \tilde{r}_o) = \rho_{so} > 0 \). Because \( \mu_s \) is empirically extremely noisy to estimate, we look for an economic prior. Given that under perfectly integrated capital markets, the Hotelling-Solow rule states that natural resource prices should grow at the world interest rate such that countries are indifferent between depletion (earning the interest rate) and keeping oil underground (earning price changes). We hence assume a risk premium on oil of zero, \( \mu_o = 0 \). Brent prices on 4 January 1982 have been 35.9 and rose to 66.6 on 21 October 2008. This amounts to a meager 2.3% return per annum over the last 26.8 years, which makes our assumption of a zero risk premium on underground oil suddenly look much more realistic. Even if we instead use the maximum oil price of 145.61, this amounts to a mere 5.3% return, which is even more in line with average money market returns.\(^4\)

How do we integrate oil wealth into a country’s budget surplus (deficit)? Let \( \theta \) denote the fraction of importance the SWF plays in the government budget. A simple way to gauge this is the following consideration. If the SWF has a size of 1 monetary unit, while the market value of oil reserves amounts to 5 monetary units, this translates into \( \theta = \frac{1}{5} \) weight for the SWF asset and \( 1 - \theta = 1 - \frac{1}{5} = \frac{4}{5} \) weight for oil revenues.\(^5\) In other words

\[ \tilde{r} = \theta \tilde{r}_s + (1 - \theta)\tilde{r}_o \]  

Note that \( 1 - \theta \) represents the implied cash holding that carries a zero risk premium and no risk in a one period consideration. Expressing returns as risk premium has the advantage that we do not need to model cash holdings. These simply become the residual asset that ensures portfolio weights add up to one without changing risk or (excess) return.

Suppose now the SWF manager is charged to maximize the utility of total government wealth rather than narrowly maximizing the utility for its direct assets under management. The optimal solution for this problem can be found from

\[ \max_w \left\{ \theta w \mu_s - \frac{\lambda}{2} \left[ \theta^2 w^2 \sigma_s^2 + (1 - \theta)^2 \sigma_o^2 + 2w \theta (1 - \theta) \rho_{so} \sigma_s \sigma_o \right] \right\} \]  

Taking first-order conditions and solving for \( w \) we arrive at the optimal asset allocation for a resource-based SWF.

\[ w^* = w_s^* + w_o^* = \frac{\mu_s}{\theta \lambda \sigma_s^2} \cdot \frac{1 - \theta \rho_{so}}{\sigma_o \sigma_s} \]  

Total demand for risky assets can be decomposed into speculative demand, \( w_s^* \), and hedging demand, \( w_o^* \).

In the case of uncorrelated assets and oil resources the optimal solution is equivalent to a leveraged (with factor \( \frac{1}{5} \)) position in the asset-only maximum Sharpe ratio portfolio or, in other words, \( w_o^* \). What is the economic intuition for this leverage? For investors with constant relative risk aversion, the optimal weight of risky assets will be independent from wealth level, which nowhere enters (4). While a given country might have little in financial wealth in the form of SWF financial assets it might be rich in natural resources and as such it requires a large multiplier. For \( \theta = \frac{1}{5} \), we would require the SWF to leverage substantially (6 times). Assuming \( \mu_s = 5, \sigma_s = 20, \lambda = 0.03 \), we get \( \frac{\mu_s}{\lambda \sigma_s} = 7.89 \approx 250\% \). The second component in Equation (5) represents hedging demand. In other words, the desirability of the risky

**Portfolio Choice for Oil-Based Sovereign Wealth Funds**

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Hedge demand is only zero if oil price risk is purely idiosyncratic. For \( \sigma_o = 40 \), \( \rho = 0.1 \), we would reduce the allocation in the risky asset according to

\[
- \frac{0.1 \times 40}{1 - 0.1} = -100\%.
\]

Positive correlation between asset and oil price risk increases the volatility of total wealth. A 100% short position in the risky asset helps to manage total risk. In case the correlation was negative, however, we would even further increase the allocation to the risky asset. The optimal position of the SWF would be 1.5 \times \text{leverage} in the global market portfolio.

While the focus of this article is not on empirical work, we should provide some indication on the oil-shock-hedging properties of traditional asset classes. Without the existence of these assets that could potentially help to reduce total wealth volatility for oil-rich investors, Equation (5) would be of little practical use. Let us look at global equities (MSCI World in USD) and U.S. government bonds (Lehman U.S. Treasury total return index for varying maturities) and oil (Crude Oil-Brent Cur Month FOB from Thompson) for the period from January 1997 to September 2008. The selection of the above-mentioned assets is motivated by some basic economic considerations. Oil tends to do well either in a political crisis (in which equities do not do well) or in anticipation of global growth (in which equities also do well). At the same time, government bonds (particularly at the long end) are a natural recession hedge and will also do well if oil prices fall. There are obviously notable exceptions. Oil and bonds will move together if an oil price increase is the cause of recession fears. In this scenario, shorter bonds should provide better returns than long bonds due to rising inflation fears. All asset classes move together if monetary loosening inflates a leverage-based bubble that drives equity and bond markets up while bonds perform due to falling interest rates.

We can endlessly talk about our economic priors, however, so we should have a look at the data instead. The results of our correlation analysis are given in Exhibit 3. In the short run (monthly data), we do not find significant correlations between oil price change and the selected asset class return. However, reducing the data frequency (i.e., increasing the period to calculate returns from) shows significantly negative correlations between oil price changes and fixed income returns. In other words, we find that long-term correlations are buried under short-term noise.

Both the degree of (negative) correlation and its significance (even though we reduce the sample size) rise as we move from quarterly to annual. Global equities,
however, provide no hedge against oil price changes. While they could still be used as a performance asset, they are of limited use as a hedge against oil price shocks.

Proponents of equity investments might suspect that we are underselling their case, as we have not been allowing for more granular equity exposures. Maybe we can identify various sectors that respond differently to oil price shocks. A global equity portfolio is already a diversified portfolio that leaves no possibility to leverage these effects. The results for this, shown in Exhibit 4, are encouraging. We find significant negative correlation for defensive consumer and health care sectors that tend to do well when the economy does badly. Results are stable and significant for different data frequencies. At the same time, the energy sector is positively related to oil and does not qualify for inclusion into SWF allocations, as we would have conjectured before.

OPTIMAL ALLOCATION BETWEEN GROWTH AND HEDGE ASSETS

The previous section has shown that traditional equities offer little hedge against oil price risks. At the same time, fixed income investments do but they do not offer the same long-term returns. A SWF manager therefore needs to extend the investment universe into performance as well as hedge assets. We remain with the setup from the previous section but extend the universe into two assets, where one asset is assumed a hedging asset (i.e., it shows negative correlation) while the second asset provides growth orthogonal to oil wealth changes. We can summarize our setup with the following distribution

$$\tilde{r}_g \sim N(\mu_g, \sigma_g^2), \tilde{r}_h \sim N(\mu_h, \sigma_h^2)$$

where $r_g$ and $r_h$ stand for the return of growth and hedge assets with $\mu_g > \mu_h$. Our correlation assumptions are

$$\text{Cov}(r_g, r_h) = \rho_{gh} \sigma_g \sigma_h > 0, \text{Cov}(r_g, \tilde{r}_h) = \rho_{gh} \sigma_g \sigma_{\tilde{h}} < 0, \text{Cov}(r_h, \tilde{r}_h) = 0$$

Note that our setup effectively assumes that an SWF can use leverage but will only go long assets. The hedge asset is negatively correlated and as such will be held in nonnegative demand. This is necessary as otherwise the assumption of nontradable wealth would become meaningless. The government budget evolves according to

$$\hat{r} = \theta \left[w_g \tilde{r}_g + w_h \tilde{r}_h \right] + (1 - \theta)\tilde{r}_o$$

where utility (i.e., risk-adjusted performance) is given by

$$u = E(\hat{r}) - \frac{1}{2} [E(\hat{r})^2 - E(\tilde{r})^2]$$

We maximize Equation (9) by setting the first-order conditions to zero and solving for $w_g, w_h$:

$$w = \frac{\mu_g - \beta_{gh} \mu_h - \rho_{gh} \sigma_g \sigma_{\tilde{h}}}{\lambda \theta (1 - \theta) \beta_{gh} \sigma_g \sigma_{\tilde{h}} - \theta (1 - \theta)}$$

### Exhibit 4

Correlation of U.S. Industry Returns with Percentage Oil Price Changes

<table>
<thead>
<tr>
<th>Frequency</th>
<th>Oil &amp; Gas</th>
<th>Basic Mats</th>
<th>Industrials</th>
<th>Consumer Gds</th>
<th>Health Care</th>
<th>Consumer Svcs</th>
<th>Telecom</th>
<th>Utilities</th>
<th>Financials</th>
<th>Technology</th>
</tr>
</thead>
<tbody>
<tr>
<td>Monthly</td>
<td>45.68%</td>
<td>5.94%</td>
<td>-12.89%</td>
<td>-18.03%</td>
<td>-13.83%</td>
<td>-21.26%</td>
<td>-1.62%</td>
<td>-6.32%</td>
<td>-18.08%</td>
<td>-11.86%</td>
</tr>
<tr>
<td></td>
<td>9.23</td>
<td>8.11%</td>
<td>-6.47%</td>
<td>-4.17%</td>
<td>-28.57%</td>
<td>-22.69%</td>
<td>-4.55%</td>
<td>-6.20%</td>
<td>-24.99%</td>
<td>-25.03%</td>
</tr>
<tr>
<td>Quarterly</td>
<td>5.96</td>
<td>8.05%</td>
<td>-0.68</td>
<td>-0.44</td>
<td>-3.11</td>
<td>-2.43</td>
<td>-0.48</td>
<td>-0.65</td>
<td>-2.69</td>
<td>-2.70</td>
</tr>
<tr>
<td>Annual</td>
<td>51.03%</td>
<td>-20.03%</td>
<td>-11.99%</td>
<td>-65.36%</td>
<td>-37.26%</td>
<td>-51.40%</td>
<td>-44.18%</td>
<td>-15.84%</td>
<td>-13.52%</td>
<td>-3.47%</td>
</tr>
<tr>
<td></td>
<td>3.14</td>
<td>-1.08%</td>
<td>-0.84</td>
<td>-4.57</td>
<td>-2.12</td>
<td>-3.17</td>
<td>-2.61</td>
<td>-0.85</td>
<td>-0.72</td>
<td>-0.18</td>
</tr>
</tbody>
</table>

Notes: The exhibit uses Dow Jones sector returns and oil (Crude Oil-Brent Cure Month FOB from Thompson) for the period from January 1982 to September 2008. This translates into 323 monthly, 109 quarterly, and 28 annual data points. For each data frequency, the first line shows the correlation coefficient, while the second line provides its t-value. We calculate t-values according to $t = \frac{\hat{r}}{\sqrt{\frac{\hat{r}^2}{n-2}}}$, where $n$ represents the number of data points and $\hat{r}$ the estimated correlation coefficient. Critical values are given by the t-distribution with $n-2$ degrees of freedom. For example, the critical value for 28 annual data points at the 95% level is 2.05. All significant correlation coefficients are gray shaded.
where $\beta_{g,h} = \frac{\omega_g \sigma_h}{\sigma_g}$, $\beta_{h} = \frac{\omega_h}{\sigma_h}$. Demand for the growth asset can be split again into speculative demand and hedging demand. Speculative demand will depend on its “alpha,” $\mu_g - \beta_{g,h} \cdot \mu_h$, versus the hedge asset (“beta”), $\beta_{h}$, adjusted excess return divided by the risk not explained by the hedge asset. Here $\rho_{g,h}$ can be interpreted as the $R^2$ of a regression of hedge versus growth asset returns. Hedging demand in turn will depend on the implicit hedging through the correlation to the hedge asset. The usual risk aversion and oil wealth importance scaling applies. A similar picture is given for the growth asset. The effect here is much more direct, however, such that the hedge demand will always be greater than for the growth asset (by definition). A clearer picture arises when we get rid of the indirect correlation by setting $\rho_{g,h} = 0$. In this case, Equations (11) and (12) become

$$w_g^* = \frac{\mu_g - \beta_{g,h} \cdot \mu_h}{\lambda \theta (1 - \rho^2_{g,h}) \sigma_g^2} \left(1 - \frac{\theta}{\mu_h} \right)$$

(13)

$$w_h^* = \frac{\mu_h - \beta_{h} \cdot \mu_h}{\lambda \theta \sigma_h^2} - \rho_{g,h} \frac{(1 - \theta)}{\sigma_h^2}$$

(14)

Now both allocations can be taken independently. The growth asset is entirely driven by its Sharpe ratio, while the hedge asset combines both speculative and hedge demand directly relating to (5). Finally we ask ourselves: How will the hedge demand (allocation of long-term, USD-denominated fixed income bonds) change as $\theta$ becomes larger, that is, as financial wealth becomes more and more dominant? In order to answer this question, we calculate $\frac{dw_h}{d\theta}$ from Equation (14):

$$\frac{dw_h}{d\theta} = \frac{-\mu_h + \lambda \rho_{g,h} \sigma_g \sigma_h}{\lambda \sigma_h^2 \theta^2} < 0$$

(15)

under the assumption that $\mu_h, \mu_g > 0, \rho_{g,h}$. Economies with falling levels of oil resources should therefore invest more like “traditional” investors with cash-like liabilities.

Again we ask ourselves which kind of investment strategies would show zero correlations with oil price movements such that we can separate investment decisions into building both a growth and a hedge portfolio. A natural candidate would be the hedge fund investments. Exhibit 5 summarizes our results. All popular hedge fund strategies we have looked at are uncorrelated to oil price movement over medium to longer term time horizons. Hedge funds could therefore be an interesting addition to investments into long government bonds.

**BACKGROUND RISK—THE IMPACT OF RESOURCE UNCERTAINTY**

There is a vast literature on background risk—the risk that is uncorrelated to the assets the investor decides upon. It exists in the background of the decision maker. Pension funds, for example, are exposed to background risk.

---

**EXHIBIT 5**

**Correlation of Hedge Fund Returns with Percentage Oil Price Changes**

<table>
<thead>
<tr>
<th>Frequency</th>
<th>Distressed Debt</th>
<th>Merger Arbitrage</th>
<th>Equity Market Neutral</th>
<th>Quantitative Directional</th>
<th>Short Bias</th>
<th>Event Driven</th>
<th>Global Macro</th>
<th>Relative Value</th>
<th>Fixed Income Arbitrage</th>
<th>Convertible Arbitrage</th>
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</thead>
<tbody>
<tr>
<td>Monthly</td>
<td>10.50%</td>
<td>9.41%</td>
<td>8.56%</td>
<td>19.65%</td>
<td>-15.19%</td>
<td>13.28%</td>
<td>23.66%</td>
<td>13.02%</td>
<td>13.39%</td>
<td>7.61%</td>
</tr>
<tr>
<td></td>
<td>1.26</td>
<td>1.13</td>
<td>1.02</td>
<td>2.39</td>
<td>-1.83</td>
<td>1.60</td>
<td>2.90</td>
<td>1.56</td>
<td>1.61</td>
<td>0.91</td>
</tr>
<tr>
<td>Quarterly</td>
<td>-7.64%</td>
<td>-19.83%</td>
<td>-15.28%</td>
<td>-12.01%</td>
<td>12.54%</td>
<td>-15.15%</td>
<td>-17.34%</td>
<td>-4.22%</td>
<td>14.12%</td>
<td>-10.86%</td>
</tr>
<tr>
<td></td>
<td>-0.53</td>
<td>-1.40</td>
<td>-1.07</td>
<td>-0.84</td>
<td>0.88</td>
<td>-1.06</td>
<td>-1.22</td>
<td>-0.29</td>
<td>0.99</td>
<td>-0.76</td>
</tr>
<tr>
<td>Annual</td>
<td>20.38%</td>
<td>6.78%</td>
<td>-22.43%</td>
<td>44.78%</td>
<td>-33.86%</td>
<td>26.77%</td>
<td>41.76%</td>
<td>28.13%</td>
<td>30.09%</td>
<td>-0.88%</td>
</tr>
<tr>
<td></td>
<td>0.75</td>
<td>0.24</td>
<td>-0.83</td>
<td>1.81</td>
<td>-1.30</td>
<td>1.00</td>
<td>1.66</td>
<td>1.06</td>
<td>1.14</td>
<td>-0.03</td>
</tr>
</tbody>
</table>

Notes: The exhibit uses HFR index returns and oil (Crude Oil-Brent Cur Month FOB from Thompson) for the period from January 1997 to September 2008. This translates into 142 monthly, 48 quarterly, and 13 annual data points. For each data frequency, the first line shows the correlation coefficient, while the second line provides its t-value. We calculate t-values according to $t = \rho \sqrt{n - 2}$, where $n$ represents the number of data points and $\rho$ the estimated correlation coefficient. Critical values are given by the t-distribution with $n - 2$ degrees of freedom. For example, the critical value for 13 annual data points at the 95% level is 2.2. All significant correlation coefficients are grey shaded.
risk in the form of mortality risk, which is independent from interest rate or equity risk.

How can we translate this idea into our framework for finding the optimal allocation for an SWF? To our knowledge this has not been addressed in the theoretical literature. So far we have assumed that \( \theta \) (i.e., the fraction of SWF assets to total sovereign wealth) is known with certainty in (5), that is, that the value of oil reserves is known to the decision maker. However, the size of an oil field is not known with great precision. Additionally, government claims are sometimes legally disputed (among neighboring countries) and new undiscovered fields might be yet to be found. Hence \( \theta \) might be best thought of as a random variable. We assume the fraction of financial wealth relative to total (financial and oil) wealth follows a uniform distribution around the government’s estimate of \( \bar{\theta} \). More precisely we assume

\[
\tilde{\theta} \sim U(\bar{\theta} - \varepsilon, \bar{\theta} + \varepsilon)
\]

(16)

It seems natural to further assume independence between background risk on the level of available oil reserves and asset risk. The joint probability density function can then be written down as

\[
f(\theta, r) = f(\theta) f(r) = \frac{1}{\sigma_a \sqrt{2\pi}} e^{-\left[ \frac{a - \bar{\theta}}{\sigma_a} \right]^2} \frac{1}{(\theta + \varepsilon) - (\bar{\theta} - \varepsilon) \varepsilon} \]

(17)

We are looking for

\[
\text{Var}(\tilde{\theta}) = E[(\tilde{\theta} - \bar{\theta})^2] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (\tilde{\theta} - \bar{\theta})^2 f(\theta, r) d\theta d\gamma
\]

in order to calculate portfolio risk. Given the joint probability density (17), this is amounts to integrating over the joint probability density where

\[
L[(\tilde{\theta}, \gamma)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (\tilde{\theta} - \bar{\theta})^2 f(\theta, r) d\theta d\gamma
\]

\[
= \frac{1}{3} \varepsilon^2 + \frac{2}{3} \bar{\theta}^2 \mu_a^2 + \frac{1}{3} \sigma_a^2
\]

(19)

The expression for (18) becomes then

\[
\text{Var}(\tilde{\theta}) = \frac{1}{3} \varepsilon^2 + \frac{2}{3} \bar{\theta}^2 \mu_a^2 + \frac{1}{3} \sigma_a^2
\]

(20)

The reader will notice that for \( \varepsilon \to 0 \) we converge to the well-known expression from undergraduate statistics, \( \text{Var}(\tilde{\theta}) = \bar{\theta}^2 \sigma_a^2 \).

What does (20) imply? As long as we have background risk in the form of uncertainty around the size of oil reserves the optimal asset allocation for the SWF (we focus on the case with uncorrelated assets and oil returns for simplicity) becomes

\[
w_r^* = \frac{1}{\lambda^2} \frac{\bar{\theta} \mu_a}{\sigma_a^2 + \varepsilon^2} + \frac{\bar{\theta} \mu_a}{\sigma_a^2 + \varepsilon^2}
\]

(21)

We now compare this with the solution in absence of background risk \( w_r^* = \frac{1}{\lambda^2} \frac{\bar{\theta} \mu_a}{\sigma_a^2} \) by building the quotient.

\[
\frac{w_r^*}{w_r} = 1 + \frac{\varepsilon^2 (\mu_a^2 + \sigma_a^2)}{\bar{\theta}^2 \sigma_a^2} > 1
\]

(22)

which will always be greater than 1. An increase in background risk will lead to a decrease in risk taking for the SWF. The effect becomes stronger the more volatile our risky asset is. Empirically we should observe that SWFs with larger resource uncertainty should invest less aggressively and vice versa. Also we would expect that economies with low reserves relative to financial wealth are less affected by resource uncertainty.

**ASSET ALLOCATION AND OIL RESERVES OVER TIME**

What will drive the optimal asset allocation for an SWF over time? How will the SWF expected to shift its assets? How fast will the financial wealth of oil-rich countries accumulate? We might want the answer these questions to either solve the dynamic portfolio choice problem for an SWF or to assess the change in global financial flows for the coming years.

A brief look at the myopic one-period solution in Equation (5) reveals that the fraction of risky assets is driven by financial wealth relative to resource wealth. For a “young” SWF where financial wealth is low relative to resource wealth, a more risky asset allocation is optimal, while mature SWFs with large assets relative to natural resources should dial back their risks. To decide on the optimal asset allocation over time, we thus need to calculate the optimal extraction policy—how fast is oil
wealth transformed into financial wealth? If expected oil price changes are high relative to asset returns (opportunity costs of keeping resources underground), we would expect a slower oil extraction and therefore a lower ratio of financial wealth to resource wealth. Also if extraction technology improves (lower extraction costs), we expect a faster “oil to equity transformation.”

We choose a standard dynamic programming framework (with a fixed time horizon) to address this question, where the optimal extraction problem is solved by recursively working backward through the well-known Bellman equation, where for \( t = 0, \ldots, n-1 \)

\[
V_t = \max \left( f_t, \xi_t - \phi \xi_t^2 \right) + \frac{1}{1+r} V_{t+1}(o_t - \xi_t) \tag{23}
\]

Here \( f_t \) represents the projected oil price for period \( t \), \( \xi_t \) stands for the level of extraction: \( f_t, \xi_t \) represents oil revenues; the state variable, \( o_t \), denotes oil reserves; and the cost function for oil extraction is assumed to be quadratic in extraction with a calibration parameter \( \phi \). In order to solve Equation (23), we also need a terminal condition: In the last period, all remaining oil wealth will be extracted (at whatever cost it takes), for \( t = n \): \( V_{nt} = f_n, o_n - \phi o_n^2 \). In other words: each period an oil extraction decision, \( \xi_t \) is made that leads to a reduction in oil reserves, \( o_t = o_{t-1} - \xi_t \), and the present value of after-cost extractions is maximized. The optimal extraction policy has to counterbalance the desire to extract all oil at once to get immediate rather than very distant cash flows against the rising extraction costs of doing so.8

We calibrate our calculations with the following sets of assumptions. Initial oil reserves in Norway are 9,947 million barrels at an oil price of USD 70/barrel. Current oil extraction is assumed to be around 720 million barrels a year. Assuming extraction is optimal (myopic) at current levels, we can calibrate the cost function parameter from \( \phi = \frac{\beta}{\delta} = 0.00000005 \). This is obviously equivalent to assuming that a (October 2008) price of 70 is equal to marginal production costs. Oil price growth is expected to be around 5% per annum, alongside a 5% risk-free rate: \( \beta = \frac{1}{1+r} - 0.9524 \). We also assume the government to be capturing 100% of revenues from oil extraction.

The results of the above calculations can be found in Exhibit 6. It is worth noting that we do not equate

---

**Exhibit 6**

Evolution of Oil Wealth (Adjusted for Extraction Costs) over Time

![Graph showing the evolution of oil wealth over time.](image)

**Notes:** Oil wealth is directly calculated from the Bellman value function; that is, we plot \( V_t \) from Equation (23) for \( t = 0, \ldots, n \) where 0 represents the year 2008. This is a better measure of wealth than simply multiplying remaining oil reserves by oil prices. After all, oil needs to be extracted first.
oil wealth with the market value of oil reserves. In fact, we argue that oil wealth is defined with the optimal extraction policy (which aims to maximize oil wealth) and therefore can be found from $V^*_0$ from Equation (23). In our example, oil wealth is assumed to be USD 572 billion instead of USD 696 billion (9,947 million barrels × current price of USD 70/barrel).

As oil wealth becomes depleted, the relative importance of financial assets to natural resources shifts. We simply calculate $\theta_t$ across time (assuming a starting position of $\theta = \frac{1}{6}$, i.e., financial wealth of USD 114.6 billion) and substitute this into Equation (5). This allows us to estimate the evolution of risky assets (as a fraction of financial wealth) over time. The results for speculative and hedging demand are given in Exhibit 7. The SWF starts out as an aggressive investment vehicle with a leveraged position (150% exposure) in the risky asset. As time goes by, hedging demand is reduced but so is speculative demand. Hedging demand is negative for positively correlated assets; that is, the SWF fund scales back risks that the investor would otherwise take on a standalone basis. With no resources left, the SWF would invest about 42% in the risky asset with the remaining allocation in cash.

The above framework can be easily applied to a multi-asset context. In this case it is trivial to expand Equations (11) and (12), while the optimal extraction policy will remain the driving force for portfolio adjustments across time.

**GOVERNANCE COSTS**

As many have observed, the biggest peril for a government-run investor is political meddling. Few SWFs are specifically set up to get rid of political influence, such as Norway’s Government Pension Fund and New Zealand’s Superannuation Fund, and many of the oil-revenue-funded SWFs operate under complete opacity. In fact, it is not uncommon to separate the management of oil revenues and sovereign assets. All this might somewhat limit the use of a normative model like the one presented in this article. However, we can use the model to put a price on ignoring the impact of underground wealth on the optimal asset allocation. We pursue this exercise by calculating the direct loss in utility from

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**EXHIBIT 7**

**Optimal SWF Allocation in Risky Asset over Time**

![Optimal SWF Allocation in Risky Asset over Time](chart)

*Notes: We assumed $\mu_a = 5$, $\sigma_a = 20$, $\lambda = 0.03$, as well as $\mu_o = 0$, $\gamma = 5$, $\sigma_o = 40$, $\rho = 0.1$. Aggregate demand for the risky asset arises from speculative and hedging demand. Over time, hedging demand reaches zero as resources become depleted.*
ignoring underground wealth for 100 different assumptions on the fraction of financial wealth to total wealth, that is, for \( \theta = 0.01, 0.02, \ldots, 1 \). All other parameters are same: \( \mu_a = 5, \sigma_a = 20, \lambda = 0.03 \) and \( \mu_o = 0, r = 5, \sigma_o = 40, \rho = -0.3 \). The optimal solutions are then plugged into the correct decision-making problem described in (4), with the difference in utility being interpreted as the security equivalent (the risk-free return that would equate both utilities).

This difference is strictly positive as seen in Exhibit 8 for all values of \( \theta \). This should not be a surprise as only the optimal solution will, by definition, maximize expected utility. Ignoring the relationship between financial assets in SWFs and total sovereign wealth leads to strong utility losses of around 600 basis points per annum for small funds and still around 400% where sovereign wealth funds represent 20% of total wealth. If all sovereign wealth is stored in financial assets, the costs of portfolio inefficiency become zero.

These inefficiency costs could be reduced by running an overlay strategy on top of financial and resource wealth. A risk-based overlay—while leaving the governance of the sovereign wealth fund and the management of oil resource revenues separate—could take the correlations between resource wealth and financial wealth into account and correct deviations from optimal asset allocation using traded derivatives.

**EXHIBIT 8**
Loss in Utility from Failing to Account for Underground Wealth

![Graph showing loss in utility from failing to account for underground wealth](image)

**CONCLUSIONS**

Sovereign wealth funds might be a new set of investors, but classic portfolio choice still applies. We found that the SWF decision-making problem can be modeled as optimal asset allocation with endowed, non-tradeable wealth. Closed-form solutions are readily available, and allocations can be separated with the usual two fund separation: the first fund being an optimal growth portfolio and the second being an oil price risk hedging portfolio. We also investigated the impact of resource uncertainty on optimal asset allocation. An SWF fund of a sovereign with considerable resource uncertainty might find it optimal to invest less aggressively than an SWF with well-established oil resources. Finally, we showed how we can model optimal asset allocation over time as a function of the optimal oil extraction policy. Maturing SWF funds will invest less aggressively, while recently funded SWFs will need to be run very aggressively to diversify total wealth.

**ENDNOTES**

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1A long time horizon does not imply low risk aversion. This is one of the most common fallacies made in asset management and usually rests with the focus on quantile-based risk management.

2See http://www.odin.dep.no/fin/engelsk/p10001617/p10002780/indexbna.html. Further information regarding the aims and policies of Norway’s State Petroleum Fund can be found in the Annual Reports, Kjaer [2001], and Norges Bank [2002].

3Other examples of portfolios funded by revenues from natural resources include the Alaska Permanent Reserve Fund (USD 23 billion), the State Oil Fund of Azerbaijan (USD 0.5 billion), Chad’s Revenue Management Fund, the National Fund of Kazakhstan (USD 1.2 billion), Venezuela’s Investment Fund for Macroeconomic Stabilisation (USD 3.7 billion), the Alberta Heritage Savings Trust Fund (CAD 3.7 billion), and the Nunavut Trust (CAD 0.5 billion). Furthermore, certain central bank funds of oil exporting countries, such as Iran, Kuwait, Oman, and Saudi Arabia, are de-facto oil revenue funds. In general, stated investment objectives are similar to those of the Norwegian fund; that is, a favorable long-term trade-off of return and risk of the financial portfolio. The risk
in aggregate wealth stemming from price changes in natural
reserves is typically ignored.

Empirical estimates of risk premia for volatile assets are
plagued with sampling error. Researchers will therefore try
to back up their assumptions not only with historical data but
also with economic theory consistent with the data. While
the Hotelling-Solow rule is ubiquitous in resource economics,
it is not unchallenged. Litzenberger/Rabinowitz [1995] and
Pagano/Pisani [2009] find theoretical and empirical (equally
sample dependent) evidence for a positive risk premium on
oil, while Trolle/Schwartz [2010] document the existence
of a variance premium. However, the results and models in
this article do not necessarily need the assumption of a zero
risk premium. Rational agents in this article decide on asset
allocation (portfolio weights) with oil wealth exogenously
given. In other words, our choice variable is not associated
with oil price growth. If we collect all terms (in our asset
allocation problem) that are a function of portfolio weights,
the expected risk premium on oil does not enter the optimi-
zation problem. The only section where this risk premium
affects the solution is Asset Allocation and Oil Reserves over
Time. For a positive risk premium, the optimal oil extraction
process would slow down (it would pay to wait a bit longer
to explore), while for a negative risk premium it would be
advisable to speed up exploration.

Alternatively one could incorporate all other items
(tax revenues, government spending, pension liabilities, etc.)
into the government budget surplus/deficit calculation as in
Doskeland [2007]. Given the relatively low volatility of these
positions, however, we will ignore this problem. It will not
change the nature of our findings.

For the more general case, we get
\[ \sigma_i^2 = \frac{1}{\lambda} \left( \frac{\mu - \theta}{\lambda \rho} \right) \left( \frac{\sigma_h^2}{\rho^2} \right), \]
however, the intuition does not change.

See Gollier [2001] for a review.

An introduction into dynamic programming is found in
Bertsekas [1976]. The interested reader can get code for
NUOPT™ for S–PLUS™ from the author on request.

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