Option Pricing and Hedging in the Presence of Cross-Hedge Risk

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Abstract
This paper addresses the question of option pricing and hedging when the underlying asset is not available for dynamic trading, and some other asset is used as a substitute. We first provide an overview of the various hedging methodologies that can be used in this incomplete market setting, distinguishing between self-financing and non-self-financing strategies. Focussing on a local risk-minimization criterion, we present an analytical expression for the optimal hedging strategy and the corresponding option price. We also provide a quantitative measure of the residual risk over the life of the option. We find that the use of the optimal strategy induces a much smaller replication error compared to the replication error induced by a naive Black-Scholes strategy, especially for low levels of the correlation between the underlying asset and the substitute. In the absence of transaction costs, we also find that cross hedge risk is more substantial than the risk induced by discrete trading for reasonable parameter values. While this result implies that trading in the substitute can only be rationalised for exceedingly high correlations, the presence of (higher levels of) transaction costs is likely, however, to make trading in the actual underlying asset a prohibitively costly alternative.

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Introduction
This paper addresses dynamic hedging of an option when the underlying asset is not available for trading, and some other asset, or portfolio, is used as a substitute. The underlying asset may be unavailable because of liquidity constraints, legal constraints, high market friction, or for other reasons. If the substitute asset were perfectly correlated with the actual underlying asset, no further risk would be introduced, since one could offset any gain or loss in the option position by dynamically trading the substitute asset. In general, however, correlation is not perfect, and the unavailability of the underlying asset induces some form of dynamic incompleteness in that perfect replication is no longer possible with a self-financing strategy. A typical example of what is known as cross-hedge risk can be found in index option markets, in which $\text{S&P100}$ options are systematically hedged using dynamic trading in $\text{S&P500}$ futures contracts, because investors cannot trade in the actual underlying asset.\(^1\)

In a similar spirit, investors are usually more willing to trade in Treasury bonds when they are dynamically hedging contingent positions on corporate bonds, mainly for liquidity reasons. Other examples include hedging a book of equity options with index futures, or hedging bond options with T-bond futures. Another example is the case of basket options; even when one can trade the basket components, one may still prefer, for efficiency reasons, to use a correlated index for pricing and hedging. More generally, one can extend the analysis to encompass the question of vega hedging an option using some other option on the same underlying asset, but with a different strike price and/or term to maturity. In this case, cross-hedge risk is induced by stochastic deformations of the term and strike structure of implied volatilities.

Given that the presence of cross-hedge risk induces a specific form of market incompleteness, it is impossible to price a contingent claim by arbitrage considerations alone. Because there is an infinite number of equivalent martingale measures (EMMs), one may simply obtain bounds on the price of a given security (see Detemple and Sundaresan (1999), who provide a binomial valuation framework for options written on non-tradable assets). In most cases, however, such bounds are too wide to be of any practical use.\(^2\) In this context, Bernardo and Ledoit (2000) and Cochrane and Saa-Requejo (2000) propose to derive tighter bounds on options on non-traded assets by imposing a restriction more stringent than the mere absence of arbitrage. In particular, Cochrane and Saa-Requejo (2000) rule out "good deals", essentially, that is, portfolios with excessively high Sharpe ratios, whereas Bernardo and Ledoit (2000) rule out "quasi-arbitrage" opportunities, that is, portfolios with a gain-loss ratio above a certain value. To further specify a unique price for the option in the absence of complete markets, one needs to introduce an optimality criterion that the strategy should satisfy. In a continuous-time setting, various criteria have been proposed in the literature.

A key distinction can be made between self-financing and non-self-financing strategies. Because of market incompleteness, self-financing strategies cannot lead to perfect replication of the option payoff, and a reasonable criterion consists of minimising the expected quadratic error. This objective has been introduced by Duffie and Richardson (1991) and has been the focus of several subsequent papers (Gourieroux, Laurent, and Pham (1998), for example), which show that the corresponding price can be expressed as an expectation of the option payoff under a so-called variance-optimal measure. Although relatively intuitive, this approach suffers from a number of shortcomings, including the absence of time-consistency and the fact that it penalises positive and negative net terminal wealth equally. To address these questions, Hodges and Neuberger (1989) have proposed to cast the problem in an expected utility framework, in which the writer of the option has preferences over his terminal net wealth. This framework naturally leads to the notion of indifference price, which is defined as a certainty equivalent, namely, the additional initial wealth that the writer of the option should be given so as to generate the same expected utility that would be achieved without the option contract. The properties of the indifference price have been investigated by Frittelli (2000) and Rouge

\(^1\) Specific assets (spread options) can be used for static hedging of cross-hedge risk.

\(^2\) See Soner, Shreve, and Cvitanic (1995) for a proof that initially buying a share of the underlying stock is the cheapest dominating policy for option replication in the presence of transaction costs.
and El Karoui (2000), who show that it can be expressed as the expectation of the option payoff under a so-called *indifference measure*. Many subsequent papers have dwelt on aspects of the computation of the indifference price (see Musiela and Zariphopoulou (2004) and Tehranchi (2004), for example). Rouge and El Karoui (2000) have shown that the limit of the indifference price as the risk aversion goes to infinity is equal to the super-replication price of the option, which is infinite in the case of a European call written on a non-traded asset.

A competing way to address the pricing and hedging problems in the presence of market incompleteness is to consider non-self-financing strategies, which are designed to perfectly replicate the option payoff at the cost of continuous cash infusions into (or withdrawals from) the replicating portfolio. Since such additions of cash are random, a risk-averse agent would rationally require that the total uncertainty involved over the remaining life of the option be minimised. This leads to the so-called global risk-minimisation criterion, introduced by Föllmer and Sondermann (1986), but global risk minimisation is not, in general, possible, which is the reason for the use of the weaker local risk-minimisation criterion (Föllmer and Schweizer (1989) and Schweizer (1991)). Instead of attempting to minimise the uncertainty with respect to cash infusion/withdrawal over the lifetime of the option, this second criterion attempts to minimise, at each date, the uncertainty over the next infinitesimal period.

In what follows, we provide a formal review of these different approaches, before focusing on local risk-minimisation, which can be shown to lead to a price equal to the limit of the indifference price for a risk aversion shrinking to zero. We also provide a quantitative measure of the residual risk over the life of the option. We find that the use of the optimal strategy induces a replication error much smaller compared to the replication error induced by a naive Black-Scholes strategy, especially for low correlation between the underlying asset and the substitute. In the absence of transaction costs, we also find that cross-hedge risk is more substantial than the risk induced by discrete trading for reasonable parameter values. Although this result implies that trading in the substitute can be rationalised only for exceedingly high correlations, the presence of (higher) transaction costs is likely to make trading in the actual underlying asset a prohibitively costly alternative.

The rest of the paper is organised as follows. Section 1 introduces the problem and derives the delta-hedging strategy of a contingent claim in the presence of cross-hedge risk. In section 2, we provide a quantitative measure of the utility loss implied by the residual risk over the life of the option and compare cross-hedge risk to the risk induced by a discrete versus continuous dynamic hedging strategy. Section 3 concludes.

### 1. Cross-Hedge Risk and Optimal Delta Hedging

The problem is to find an optimal dynamic hedging strategy for a standard European call option written on some risky underlying asset $X$ when this asset $X$ is *not* available for trading. Therefore, for hedging purposes, one must use a substitute. The set of possible investment opportunities for the investor includes one risky, traded asset, denoted by $S$, which, *ex ante*, is imperfectly correlated with the non-traded underlying asset $X$. In addition, we assume that there is a risk-free asset, with a constant interest rate denoted by $r$. Uncertainty in the economy is represented through a standard probability space $(\Omega, \mathcal{F}, \mathbb{P})$ supporting two Brownian motions $z^X$ and $z^S$ with correlation coefficient $\rho$. The finite time span is represented by the interval $[0; T]$, where $T$ is the maturity date of the option. We assume that the information set to which the investor has access at time $t$ is $\mathcal{F}_t$, in which the family $(\mathcal{F}_t)_{t\leq T}$ is the augmented version of the filtration generated by $z^X$ and $z^S$. Under the subjective probability measure $\mathbb{P}$, the price of the traded substitute asset
and the non-traded underlying asset are assumed to evolve as:

\[ dS_t = S_t [\mu_S dt + \sigma_S dZ^S_t] \]

\[ dX_t = X_t [\mu_X dt + \sigma_X dZ^X_t] \]  

(1.1)

All parameters are assumed to be constant. As a consequence, the market price of stock risk, \( \lambda_S = \frac{\mu_S - r}{\sigma_S} \), is also constant. Given our dynamics (1.1), it has been shown by Harrison and Kreps (1979) that the market is (dynamically) complete if and only if the absolute correlation |\( \rho \)| of the two sources of uncertainty is equal to 1. Of course, this condition is satisfied if X is traded.

We think of an investor writing the option, receiving a premium and dynamically hedging the promised payoff. If the market were complete, there would be a unique fair price for the option, i.e., a unique price consistent with the absence of arbitrage opportunities. As shown by Harrison and Kreps (1979), the completeness entails the uniqueness of the equivalent martingale measure (EMM), under which the discounted stock price \( e^{-rt}S_t \) follows a martingale. The option payoff can be perfectly replicated by a dynamic self-financing trading strategy in the stock, and its price is the value of the replicating portfolio. Since the discounted value of this portfolio is a martingale under the EMM, the price of the option is uniquely determined as the expectation of the discounted payoff under the EMM.\(^3\) When |\( \rho \)| is different from 1; however, the market is incomplete, i.e., the payoff can no longer be perfectly replicated with a self-financing strategy. Indeed, for any trading strategy there remains a residual risk stemming from the imperfect correlation of the Brownian motions \( Z^S \) and \( Z^X \). As a result, the absence of arbitrage entails the existence of infinitely many EMMs. To assign a price to the option one must choose an EMM. As explained in the introduction, the EMM will be chosen according to a risk minimisation criterion.

1.1 Option Pricing and Hedging in Incomplete Markets with Non-Self-Financing Strategies

The writer of the option receives a premium, which will be invested in the risky traded asset and in cash. We denote by \( n^S_t \) the number of shares of stock held in the portfolio at time \( t \) and with \( A_t \) the value of the wealth process, with the wealth \( A_0 \) at time 0 given by the premium, augmented by the initial wealth, if any, in possession of the seller of the option. Over a small interval \([t, t + dt]\), the wealth process evolves as:

\[ dA_t = n^S_t dS_t + (A_t - n^S_t S_t) r dt + dC_t \]  

(1.2)

where \( dC_t \) is an amount of money infused in the portfolio (\( dC_t < 0 \) describes a withdrawal). (1.2) defines \( C_t \) up to the constant \( C_0 \) which is taken equal to the initial amount invested, \( A_0 \). \( C_t \) is thus the total amount of money that has been brought by the investor from date 0 to date \( t \): hence it is natural to call \( C \) the cost process associated with the strategy. A self-financing strategy is characterised by \( dC_t = 0 \): it requires an initial capital, and no cash infusion after date 0. We consider only strategies that generate exactly the option payoff \( (X_T - K)^+ \) at maturity \( T \).\(^4\)

The quantity \( \int_t^T e^{-r(t-s)} dC_s \) is the discounted amount of money that needs to be infused in (or withdrawn from) the portfolio between dates \( t \) and \( T \) to deliver the option’s payoff at maturity. As seen from date \( t \), this quantity is uncertain, so a risk-averse agent will focus on minimising the associated \( \text{ex ante} \) aggregate risk, namely:\(^5\)

\[ R_t = \mathbb{E}_t^P \left[ \left( \int_t^T e^{-r(t-s)} dC_s \right)^2 \right] \]  

(1.3)

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3 - It is easy to verify that, in this case, we have \( X_t = X_0 e^{\left[ (\mu_S - r) - \frac{1}{2} \sigma_S^2 \right] t + \sigma_S Z^S_t} \) with \( \rho = -1 \) or 1. Hence a European call on X is essentially a polynomial option on \( S \). The price of the option and the replicating strategy can then be obtained through the Black-Scholes formula. See Macovski and Quittard-Pinon (2006).

4 - Such strategies do always exist: it suffices to take any process \( \int_t^T e^{-r(t-s)} dC_s \) and to make the balance at maturity.

5 - It can be shown that the expected cost \( \mathbb{E}_t^P \left[ \int_t^T e^{-r(t-s)} dC_s \right] \) is zero, at least for optimal risk-minimising strategies, which justifies the sole focus on the second moment of the cumulative cost process.
This criterion was introduced by Föllmer and Sondermann (1986), and can be formally written as:

\[
\text{For all } t, \min_{(\phi, A)} \mathbb{E}_t^R; \text{ subject to } A_T = (X_T - K)^+ \quad (1.4)
\]

Equivalently, one can choose \((\phi, A)\), where \(\phi = n^S_t\) as the pair of control variables: \(\phi_t\) is thus the amount invested in the stock at time \(t\), and (1.2) can be rewritten as:

\[
dA_t = \phi_t \frac{dS_t}{S_t} + (A_t - \phi_t) \, r \, dt + dC_t
\]

Solutions to (1.4) are called \textit{global risk-minimising} strategies. As pointed out by Föllmer and Schweizer (1990), problem (1.4) can always be solved when the Sharpe ratio \(\lambda_S\) is zero. This assumption means that the discounted value process \((e^{-r t} S_t)\) already follows a martingale under \(\mathbb{P}\), so that \(\mathbb{P}\) is an EMM. But when \(\mathbb{P}\) is not an EMM, this problem may have no solution. Hence, a weaker criterion has been introduced, which is known as \textit{local risk minimisation} (Föllmer and Schweizer (1989)), which refers to a myopic programme in which the investor seeks to minimise, at each date, the expected squared error of the cost over the next infinitesimal period. An intuitive description of this objective can be given in a discrete-time model, where the expected cost over the next period can be expressed as:

\[
\mathbb{E}_t^\mathbb{P}[(\Delta C_t)^2] = \mathbb{E}_t \left[ (A_{t+\Delta t} - A_t - n^S_t(S_{t+\Delta t} - S_t) - (A_t - n^S_t S_t)(e^{r(t+\Delta t)} - e^{r t}) )^2 \right]
\]

The idea is then to minimise this quantity at each date, with respect to \((n^S_0, n^S_{\Delta t}, \ldots, n^S_{t+\Delta t})\), which gives rise to the notion of a locally-risk-minimising strategy. In discrete time (where \(\Delta t\) is normalised to 1), this is a workable objective, and minimisation can be achieved by backward induction (Föllmer and Schweizer (1989)). In continuous time (where \(\Delta t\) shrinks to zero), the precise definition and the computation of a locally risk-minimising strategy is more subtle, and requires formalism beyond the scope of this paper (see Schweizer (1990) and Schweizer (1991) for more detail). The following proposition provides, under the assumptions of our model, an explicit characterisation for the optimal locally risk-minimising strategy, as well as for the option price, which is given by the discounted value of the option payoff under a so-called minimal entropy measure.

\textbf{Proposition 1}

The locally risk-minimising strategy is described by the following elements.

- The investment in the stock is:

\[
\hat{n}^S_t = \frac{\sigma_X X_t}{\sigma_S S_t} \rho c_X(t, X_t) = \frac{\sigma_X X_t}{\sigma_S S_t} \rho e^{(\mu_X - r - \rho \sigma_X \lambda_S)(T-t)} \mathcal{N}(d_{1,t})
\]

where \(c(t, X_t)\) is the minimal entropy price:

\[
c(t, X_t) = e^{-r(T-t)} \mathbb{E}_t^\mathbb{Q}[(X_T - K)^+]
= e^{(\mu_X - r - \rho \sigma_X \lambda_S)(T-t)} X_t \mathcal{N}(d_{1,t}) - K e^{-(T-t)} \mathcal{N}(d_{2,t})
\]

In these equations, \(\mathbb{Q}\) is the minimal martingale measure in the sense of Föllmer and Schweizer (1990), defined as:

\[
\left. \frac{d\mathbb{Q}}{d\mathbb{P}} \right|_t = e^{-\frac{1}{2} \lambda_S^2 (T-t) - \lambda_S (\sigma^2_S - z^S_t)}
\]
and:

\[ d_{1,t} = \frac{1}{\sigma_X \sqrt{T-t}} \left[ \ln \frac{X_t}{K} + \left( \mu_X - \rho \sigma_X \lambda_S + \frac{\sigma_X^2}{2} \right) (T-t) \right] \]

\[ d_{2,t} = d_{1,t} - \sigma_X \sqrt{T-t} \]

- The investment in cash is:

\[ c(t, X_t) - \hat{n}_t^S S_t \]

If the Sharpe ratio of the traded substitute is equal to zero, the minimal martingale measure coincides with the original measure \( \mathbb{P} \), and the above strategy is globally-risk-minimising.

**Proof.** See appendix A.1.

The following proposition gives an expression for the “residual” aggregate risk, i.e., the aggregate risk that remains when the seller of the option follows the above strategy.

**Proposition 2**

The aggregate risk when the writer of the option implements the locally-risk-minimising strategy at time \( t \) is equal to:

\[ R_t^{\text{lr}} = \sigma_X^2 (1 - \rho^2) \int_t^T e^{-2r(s-t)} \mathbb{E}^{\mathbb{P}} \left[ X_s^2 c_X(s, X_s)^2 \right] ds \]

This can be approximated by:

\[ R_t^{\text{lr}} \approx \sigma_X^2 (1 - \rho^2) c_X(0, X_0)^2 X_0^2 \frac{1 - e^{-2r(T-t)}}{2r} \]

**Proof.** See appendix A.1.

In line with the intuition, equation (1.7) shows that the residual risk is increasing in \( \sigma_X \), the underlying asset volatility, in the initial delta, which represents the option sensitivity to a given change in the underlying asset price, and in the time to maturity \( T \). If the two assets are perfectly correlated, then we have \(|\rho| = 1\) and the aggregate risk cancels out, as it should, since the locally-risk-minimising strategy is self-financing when the market is complete. It is interesting to note that this aggregate risk indicator does not depend on the parameters (volatility and Sharpe ratio) of the traded substitute.

1.2 Option Pricing and Hedging in Incomplete Markets with Self-Financing Strategies

Here we assume that the writer of the option follows a self-financing strategy \((\phi_t)_{t \geq 0}\), so the value of his replicating portfolio evolves as:

\[ dV_t = [rV_t + \phi_t \sigma_S \lambda_S] dt + \phi_t \sigma_S dZ_t^S \quad (1.8) \]

Here, we let the value of the replicating portfolio be \( V \), to emphasise the difference from the case of non-self-financing strategies, in which the wealth process was denoted \( A \). In incomplete markets, no self-financed strategy can generate the option payoff with probability 1 at maturity. Hence the writer is left with a terminal surplus (or deficit) \( V_T - (X_T - K)^+ \) at time \( T \). A measure of the gap risk induced by the previous self-financing strategy is given by the expected squared value of the terminal surplus. In fact, as shown in the following proposition, there is a relationship between quadratic error for the self-financing strategy and the aggregate risk indicator \( R_0 \) for the corresponding non-self-financing strategy at time 0 (see (1.3)), up to a scaling factor.

**Proposition 3**

Let us consider an investment process \( \tilde{\phi} \), the associated self-financing strategy with value process \( V \) given by (1.8), and a non self-financing strategy characterised by \( \phi \) and its cost process \( C \), with value process \( A \) given by (1.5) and satisfying \( A_T = (X_T - K)^+ \) and \( A_0 = V_0 \). Then we have:

\[ R_0 = e^{-2rT} \mathbb{E}^{\mathbb{P}} \left[ (V_T - (X_T - K)^+)^2 \right] \]
Proof. See appendix A.3.
Together with the expression of the aggregate risk of the locally-risk-minimising strategy (see proposition 2), this proposition gives in turn an expression for the quadratic error induced by the self-financing version of this strategy. This proposition also provides a risk measure that can be applied consistently to self-financing and to non-self-financing strategies. In what follows, we will simply refer to it as the risk or the replication error induced by the strategy.

We now assume that the writer of the option has constant absolute risk aversion (CARA) preferences over the net final wealth $V_T-(X_T-K)^+$. The following proposition gives the optimal asset allocation strategy, that is, the strategy $(\phi^*_t)_{0 \leq t \leq T}$ that achieves the maximum expected utility of terminal surplus for the seller of the option. This programme is formally written as:

$$\max_{(\phi_t)_{t \in [0,T]}} \mathbb{E}_P^\phi \left[-e^{-\alpha(V_T-(X_T-K)^+)} \right]$$

(1.9)

Let $\tilde{\phi}$ be its solution, and $\phi^0$ the optimal strategy when no option is written. As in Musiela and Zariphopoulou (2004), we then define the indifference hedging strategy as:

$$\phi^{*,\alpha} = \tilde{\phi} - \phi^0$$

**Proposition 4**

*The solution to (1.9) is given by:*

$$\tilde{\phi}_t = \frac{e^{-r(T-t)} \lambda_S}{\alpha} + \frac{\rho X_t \sigma_x}{\sigma_S} b_X(t, X_t), \quad t \leq T$$

and the indifference hedging strategy is:

$$\phi^{*,\alpha}_t = \frac{\rho X_t \sigma_x}{\sigma_S} b_X(t, X_t), \quad t \leq T$$

(1.10)

where $b(t, X_t)$ is the Hodges and Neuberger (1989) indifference selling price of the option. $b$ is the solution to a quasi-linear PDE given in appendix A.4, equation (A.11).

**Proof.** See appendix A.4.

The interpretation of the indifference price (Hodges and Neuberger (1989)) is as follows: suppose that at time $t$ the investor has some financial wealth $a$. He is then indifferent between the following two strategies: (a) sell the option for $b(t, X_t)$, invest $A_0 = a + b(t, X_t)$ in the optimal asset strategy (1.10) and pay $(X_T - K)^+$ at time $T$; (b) write no option and invest $a$ in the strategy that is optimal if no option is written). Interestingly, for CARA preferences the indifference price does not depend on initial wealth $a$.

From proposition 4 we obtain a three-fund separation theorem, with the optimal demand for the risky asset being the sum of two components, the standard speculative demand $\frac{e^{-r(T-t)} \lambda_S}{\alpha} \sigma_S$ and the hedging demand against the non-traded underlying, $\phi^{*,\alpha}_t$.

**Proposition 5**

*The indifference price of the option is given by:

$$b(t, X_t) = \frac{e^{-r(T-t)}}{\alpha} \ln \mathbb{E}_t^P \left[e^{\alpha(1-\rho)(X_T-K)^+}\right]$$

(1.11)
where \( \widetilde{\mathbb{P}} \) is the indifference measure, whose density with respect to \( \mathbb{P} \) is given by:

\[
\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}}|_t = e^{-\frac{1}{2} \rho^2 \lambda^2 T (T-t) - \rho \lambda s (\Phi^X_T - \Phi^S_T)}
\]

Equivalently, the indifference price can be written in terms of the minimal martingale measure, \( \mathbb{Q} \):

\[
b(t, X_t) = \frac{e^{-r(T-t)}}{\alpha(1-\rho^2)} \ln \mathbb{E}_t^{\mathbb{Q}} \left[ e^{\alpha(1-\rho^2)(X_T - K)^2} \right]
\] (1.12)

Proof. See appendix A.5.

We can now state the following result, which is a special case of a proposition established by Rouge and El Karoui (2000).

**Proposition 6**

When \( \alpha \) shrinks to zero, the indifference price \( b(t, X_t) \) (resp. the indifference delta \( b_X(t, X_t) \)) converges to the minimal entropy price \( c(t, X_t) \) (resp. to the minimal entropy delta \( c_X(t, X_t) \)).

Proof. See appendix A.5.

This last result provides some rational justification for the somewhat heuristic local risk minimisation criterion, which leads to the same solution as the expected utility maximisation objective under the assumption of a vanishing risk aversion.

1.3 The Complete Market Case

We now examine the complete market case, in which \( S \) and \( X \) are pathwise equal (almost surely). In this particular case, it can be shown that the hedging strategy agrees with the standard Black-Scholes delta-neutral dynamic hedging strategy. This is the content of the following proposition. 7

**Proposition 7**

Suppose that \( S = X \) pathwise almost surely. Then:

- The indifference price \( b(t, X_t) \) coincides is equal to the Black-Scholes price, i.e.
  \[
b(t, X_t) = X_t \mathcal{N}(d_{1,t}^{\text{comp}}) - K e^{-r(T-t)} \mathcal{N}(d_{2,t}^{\text{comp}})
\]
  where \( d_{1,t}^{\text{comp}} = -\frac{1}{\sigma_X \sqrt{T-t}} \left[ \ln \frac{X_t}{K} + \left( r + \frac{\sigma_X^2}{2} \right) (T-t) \right] \) and \( d_{2,t}^{\text{comp}} = d_{1,t} - \sigma_X \sqrt{T-t} \);

- The hedging strategy \( \phi_{\text{hed}}^{\text{comp}} \) coincides with the standard Black-Scholes delta neutral strategy, that is, the number of shares of \( X \) to be held in the hedging portfolio is equal to \( \mathcal{N}(d_{1,t}) \).

Proof. See appendix A.6.

We now conduct a series of numerical experiments that attempt to assess the economic significance of cross-hedge risk.

2. Numerical Assessment of Cross-Hedge Risk

The risk measure that we use is the one given by proposition 3. Our base-case parameters are displayed in table 1. It should be noted that we have assumed that Sharpe ratios, volatilities and initial values are identical for the substitute asset and the true underlying, so that we can obtain the case where \( S \) and \( X \) are (almost surely) pathwise equal in the limit of \( \rho \) converging to 1. As a first numerical assessment of cross-hedge risk, we present the risk indicator \( \sqrt{\rho_{\text{emp}}} \) normalised by the option price for the locally risk-minimising strategy as a function of the correlation \( \rho \) (see figure 1). These results suggest that cross-hedge risk is a substantial source of error in option

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7 It is a general result that in a complete market case the indifference price agrees with the unique no-arbitrage price (see Rouge and El Karoui (2000)).
replication. In fact, replication error levels are a large fraction (sometimes greater than 100%) of the minimal entropy price for the option. Consistent with proposition 2, the replication error is increasing both in the volatility of the underlying and in the maturity of the option; we also find that it is a decreasing function of the correlation of the substitute and the underlying asset, and that it converges to 0 in the limit of perfect correlation between the actual underlying asset and the substitute asset, as expected. After confirming the economic significance of cross-hedge risk, we next run a series of numerical experiments in an attempt to compare the performance of the optimal locally-risk-minimising strategy and that of two heuristic strategies based on the Black-Scholes delta, and trading in the substitute in continuous time and in the actual underlying in discrete time respectively.

2.1 Comparison of the Optimal Strategy and a Continuous-Time Heuristic Strategy
The heuristic strategy is based on the assumption that the option can be hedged considering the delta derived from the Black-Scholes formula. More specifically, we assume that, at each time period, the following amount \( \phi_{t}^{\text{heur}} \) is invested in the replicating asset, with the remaining amount of wealth invested in cash:

\[
\phi_{t}^{\text{heur}} = S_{t} \mathcal{N}(d_{1,t}^{\text{comp}})
\]

where \( \mathcal{N}(d_{1,t}^{\text{comp}}) \) is the Black-Scholes delta computed using the parameters of the underlying asset (see proposition 7). In the numerical exercise, the initial wealth \( A_{0}^{\text{heur}} \) is taken to be the initial minimal entropy price of the option to allow for a fair comparison. Let \( A_{T}^{\text{heur}} \) be the payoff generated by this heuristic strategy. The associated risk is thus defined as:

\[
R_{0}^{\text{comp}} = e^{-2rT} \mathbb{E} \left[ \left( A_{T}^{\text{heur}} - (X_{T} - K)^{+} \right)^{2} \right]
\]

which is to be compared to the risk \( R_{0}^{\text{nm}} \) incurred by the locally-risk-minimising strategy.9 The ratio \( \sqrt{R_{0}^{\text{heur}}}/\sqrt{R_{0}^{\text{nm}}} \) is plotted in figure 2 for various levels of the correlation and three different volatilities X of the underlying asset. Given our base-case parameter values (see table 1), we see from proposition 7 that the locally risk-minimising strategy converges to the Black-Scholes strategy as \( \rho \) grows to 1. In particular, both \( R_{0}^{\text{nm}} \) and \( R_{0}^{\text{heur}} \) are zero when there is perfect correlation.9 Figure 2 shows that the naive strategy induces an error much larger than the replication error implied by the locally-risk-minimising strategy, especially for low correlation. For such low or medium correlations (below 50%), the increase in the error as a result of implementing the heuristic strategy rather than the locally-risk-minimising strategy is also growing in the volatility of the underlying.

2.2 Comparison of the Optimal Strategy and a Discrete-Time Heuristic Strategy
In the derivation of the locally-risk-minimising strategy (see proposition 1), we have assumed that the true underlying was not available for continuous trading and that an imperfectly correlated substitute was used for hedging purposes. In this subsection, we attempt to compare the risk of the resulting strategy and the risk incurred by a discrete-time strategy using the actual underlying asset. The discrete-time strategy that we consider here is a heuristic strategy inspired by the one proposed in Leland (1985).10 This is a non-self-financing strategy in which the replicating portfolio is rebalanced at regular dates \( 0 = t_{0} < t_{1} < \ldots < t_{n-1} < t_{n} = T \) according to the following rule:

\[
n_{t_{i}}^{S} = BS_{X}(t, X_{t_{i}}) \quad A_{t_{i}}^{\text{disc}} = BS_{X}(t, X_{t_{i}})
\]

8 - It should be emphasised that there is no reason ex ante for the latter to yield a smaller error than the heuristic one, because in contrast to a globally risk-minimising strategy, it has not been designed to minimise \( R_{0} \).

9 - Since the two strategies are equivalent for \( \rho \) growing to 1, the ratio of errors should in theory converge to 1. Although this convergence is clear from figure 2 for \( T = 3 \) months, it is less obvious for \( T = 1 \) year. Rounding errors that occur when dividing a small quantity by another small quantity, combined with sampling errors incurred by the Monte-Carlo method, are the reason for this observation.

10 - For simplicity, we report the results without implementing the adjustment to the volatility proposed by Leland (1985). We obtain qualitatively similar results when the volatility adjustment is incorporated.
where $BS(t, X_t)$ is the Black-Scholes price of the option at time $t$, namely:

$$BS(t, X_t) = X_t e^{-r(T-t)} N(d_{2,t}^{\text{comp}}) - Ke^{-r(T-t)} N(d_{1,t}^{\text{comp}})$$

Over each sub-interval $[t_i, t_{i+1}]$, the replicating portfolio is self-financing and evolves as:

$$dA_t^{\text{disc}} = n^S_t dS_t + (A_t^{\text{disc}} - n^S_t S_t) e^{(r-t) t} dt, \quad t_i \leq t < t_{i+1}$$

On the dates $t_i$ for $i = 1, \ldots, n$, the process $A^{\text{disc}}$ may exhibit left discontinuities, and the writer of the option must bring an amount $\Delta C_{t_i}$ to have the value of his portfolio equal to $BS(t_i, X_{t_i})$, where:

$$\Delta C_{t_i} = BS(t_i, X_{t_i}) - A_{t_i}^{\text{disc}}$$ \tag{2.2}$$

The risk of the discrete-time strategy is again given as the expected squared aggregate cost:

$$R_0^{\text{disc}} = \mathbb{E}^P \left[ \sum_{t_i} e^{-r t} \Delta C_{t_i} \right]^2$$

which is to be compared to $R_0^*$, the risk incurred by the locally-risk-minimising strategy. Of course, the strategy described in (2.1) is not meant to be optimal in any sense, and is purely heuristic. In particular, there is no particular reason, except for simplicity and tractability, to choose the Black-Scholes price as a reference price. As in the continuous time case (see proposition 3), one can show that this risk is equal to the expected quadratic error of an otherwise identical self-financing strategy, up to a factor $e^{-2rT}$:

**Proposition 8**

Consider the self-financing strategy with value process $V_0^{\text{disc}}$ such that $V_0^{\text{disc}} = A_0^{\text{disc}}$, the strategy is buy-and-hold over each interval $[t_i, t_{i+1}]$, and the investor buys $n^S_t$ shares of $S$ at time $t_i$. Then we have:

$$R_0^{\text{disc}} = e^{-2rT} \mathbb{E}^P \left[ (V_T^{\text{disc}} - (X_T - K))^2 \right]$$

**Proof.** See appendix A.7.

One can also provide an approximation of this discrete error as follows.

**Proposition 9**

Assume that the true underlying is used in the replicating portfolio, and that rebalancing takes place at discrete dates $t_i = \frac{iT}{N}, i = 0, \ldots, N - 1$ according to the delta-hedging rule. The expected quadratic error is:

$$R_0^{\text{disc}} \approx \frac{1}{2} \sigma^4_X (\Delta t)^2 \sum_{i=0}^{N-1} \mathbb{E} \left[ \gamma^2_{t_i} X_{t_i}^4 \right]$$

where $\gamma_{t_i}$ is the Black-Scholes gamma of the option, i.e., $\gamma_{t_i} = BS_{XX}(t_i, X_{t_i})$. This can be approximated as:

$$R_0^{\text{disc}} \approx \frac{1}{2} \sigma^4_S \frac{1}{n} \left. \sigma^4_X X^4_0 \right| (d_{1,0}^{\text{comp}})^2 \Delta t$$

where $n$ is the standard normal density function.

**Proof.** See appendix A.8.

The approximation of proposition 9 shows that the error induced by the discrete strategy depends only on the maturity $T$ through the $n(d_{1,0})$ factor. Hence one can expect $R_0^{\text{disc}}$ to be relatively little sensitive to the parameter $T$, which will be confirmed by numerical inspection. We now turn to the following question: if the underlying asset is available for trading but at low frequency (monthly frequency for hedge funds, for example), how well correlated should the substitute asset be to justify using it in a higher frequency delta-neutral hedging strategy?
To answer this question, and to compare the errors $R_{0}^{\text{rm}}$ and $R_{0}^{\text{disc}}$, we compute break-even correlations $\rho_{\text{bc}}$, defined as follows:

$$R_{0}^{\text{rm}} = R_{0}^{\text{disc}} \quad \text{for} \quad \rho = \rho_{\text{bc}}$$

Hence, when the correlation $\rho$ is lower than the break-even $\rho_{\text{bc}}$, the locally-risk-minimising strategy involving continuous trading in the substitute generates a replication error $R_{0}^{\text{rm}}$ larger than the risk implied by a discrete trading strategy in the true underlying. Figure 3a presents the risk associated with the discrete strategy, normalised by the Black-Scholes price of the option. Overall, the replication error induced by discrete trading appears to be smaller than that of the replication error induced by cross-hedge risk. This finding is confirmed in figure 3b, which displays the break-even correlation as a function of the trading interval $\Delta t$. We find that trading in the substitute can be rationalised only for exceedingly high correlation. In other words, it appears that cross-hedge risk is more substantial than the risk induced by discrete trading for reasonable parameter values. We also find that the break-even values are higher for the one-year option than for the three-month one. This can be explained by the fact that the error induced by the discrete strategy is much less sensitive to the option maturity than is the error induced by cross-hedge risk (see the approximation for $R_{0}^{\text{rm}}$ in proposition 2, with a quantity $(1 - \rho_{\text{bc}}^2)TC_X(0, X_0)^2$ that does not vary substantially across maturities).

2.3 Introducing Transaction Costs

We have so far assumed away the presence of transaction costs. This assumption is not consistent, however, with the fact that one of the typical reasons a substitute is used in place of the true underlying asset has to do, as it happens, with the substantially higher transaction costs associated with trading in the true underlying of the option. Intuitively, introducing such costs will lead to decreasing the performance of the discrete-time strategy, and will make trading in the substitute a more attractive alternative. In what follows, we make the simplifying assumption that trading in the substitute can be done in continuous-time at virtually no cost, while we assume that trading in the true underlying asset generates proportional transaction costs (Leland (1985)). Each adjustment of the replicating portfolio is therefore assumed to generate the following amount of transaction costs:

$$TC_{t_i} = \frac{k}{2} \left| n_{t_i}^S - n_{t_{i-1}}^S \right| S_{t_i}$$

where $k$ is the magnitude of transaction costs, assumed to be proportional to the traded amount. Hence, for the writer of the option, the total cost incurred at time $t_i$ is:

$$\Delta C_{t_i} = BS(t_i, X_t) - A_{t_{i-1}}^{\text{disc}} + TC_{t_i}$$

and we consider the aggregate discounted cost $\sum_i e^{-r t_i} \Delta C_{t_i}$. Figures 4a and 4b present the average value and the variance of the aggregate cost, normalised by the initial Black-Scholes price of the option, $BS(0, X_0)$. As was to be expected, the average aggregate cost is an increasing function of $k$, and decreases when adjustments of the portfolio are less frequent. It can be noted that transaction costs represent more than the option price on average if weekly trading is implemented, especially if the option has a long term to maturity. For monthly trading, they still represent a significant 20% or more of the option price. These results shed new light over the previous finding that trading in the substitute can be rationalised only for exceedingly high correlations, which were obtained under the assumption of no transaction costs for the true underlying asset. In fact, the introduction of transaction costs leads not only to increasing the average aggregate cost but also to an increase in the uncertainty over aggregate cost, as can be seen from figure 4b. Overall, our results confirm that trading in the actual underlying asset can be prohibitively costly, so trading in the substitute may be a legitimate alternative, even though it implies a very substantial replication error.

11 - We also find that the replication error expressed as a percentage of the option price is higher for the three-month option than for the one-year option. This can be explained by the fact that the replication error $R_{0}^{\text{rm}}$ increases only slightly when the maturity increases from three months to one year (as shown by proposition 9), and this change is small compared to the increase in the Black-Scholes price.

12 - Here we show the variance, as opposed to the second moment, of the aggregate cost. The second non-centred moment is not easily interpreted in the presence of transaction costs because the first moment is no longer zero.
3. Conclusion
In this paper, we provide an overview of the pricing and hedging methodologies that can be used in the presence of cross-hedge risk, and discuss some implications. Using a local-risk-minimisation criterion, we present an analytical expression for the optimal hedging strategy and the corresponding option price. We also provide a quantitative measure of the residual risk over the life of the option. We find that the use of an optimal locally-risk-minimising strategy induces a replication error much smaller than the replication error induced by a naive Black-Scholes strategy, especially for low correlation of the underlying asset and the substitute, even though the replication error remains large when compared to the option price. We also find that in the absence of transaction costs, cross-hedge risk is more substantial than the risk induced by discrete trading for reasonable parameter values. Although this result implies that trading in the substitute can be rationalised only for exceedingly high correlation, the presence of (higher) transaction costs is likely to make trading in the actual underlying asset a prohibitively costly alternative.
A. Proofs of the Propositions

A.1 Proof of Proposition 1

Let \( \tilde{S}_t = e^{-rt} S_t \) be the discounted value of the traded substitute at time \( t \). This process follows a martingale under the martingale measure \( \mathbb{Q} \), since we have:

\[
d\tilde{S}_t = \tilde{S}_t \sigma_S \, dz^S_t
\]

where \( dz^{S,Q}_t \equiv dz^S_t + \lambda_S \, dt \) is the increment to a \( \mathbb{Q} \)-Brownian motion. Hence we can write the Kunita-Watanabe decomposition of the discounted option payoff under \( \mathbb{Q} \):

\[
e^{-rT} (X_T - \mathcal{K})^+ = H_0 + \int_0^T \xi_t \, d\tilde{S}_t + L^H_T \tag{A.1}
\]

where \( L^H \) is a \( \mathbb{P} \)-martingale orthogonal to \( \tilde{S} \) under \( \mathbb{Q} \).\(^{13}\) Levy’s theorem shows that the process \( B \) defined by \( dB_t = \frac{dz^S_t - \lambda_S dt}{\sqrt{2 \lambda_S}} \) is a \( \mathbb{P} \)-Brownian motion and that it is independent of \( z^S \). Then, by Girsanov’s theorem, \( (z^S, \xi, B) \) also follows a \( \mathbb{Q} \)-2-dimensional Brownian motion. Since \( L^H \) is a martingale under \( \mathbb{Q} \) and is orthogonal to \( \mathcal{F} \), the martingale representation theorem shows that we have \( L^H_t = \psi_t dB_t \) for some process \( \psi_t \). In particular, \( L^H \) is orthogonal under \( \mathbb{P} \) to the martingale part of \( \tilde{S} \), where the martingale part of \( S \) under \( \mathbb{P} \) is defined as \( G_t = \int_0^t \sigma_S S_s \, dz^S_s \).

Now let us \( P_t = e^{-r(T-t)} \mathbb{E}^Q_t [(X_T - \mathcal{K})^+] \). Using (A.1) we obtain that:

\[
P_t = e^{rt} \left[ H_0 + \int_0^t \xi_s \, d\tilde{S}_s + L^H_t \right] \tag{A.2}
\]

Consider now the non-self-financing strategy with value \( \tilde{A}_t = P_t \) and number of shares of \( S \) given by \( \tilde{n}^S_t = \xi_t \). Given (1.2) and (A.2), we obtain that \( d\tilde{C}_t = e^{rt} \, dL^H_t \). This shows that, as \( L^H \), \( \tilde{C} \) is a \( \mathbb{P} \)-martingale orthogonal to \( \tilde{S} \). As shown by Schweizer (1991), a strategy \( (A, n^S) \) is locally risk-minimising if and only if the associated cost process follows a \( \mathbb{P} \)-martingale orthogonal to \( \tilde{S} \). Hence the strategy \( (\tilde{A}, \tilde{n}^S) \) is locally risk-minimising.

We now provide an explicit expression for the random variable \( P_t \), which is called the minimum entropy price of the option. The Black-Scholes formula implies that:

\[
P_t = c(t, X_t) = e^{(\mu x - r - \sigma^2 \rho \lambda S)(T-t)} \left[ X_t \mathcal{N}(d_1, t) - K e^{(\mu x - r - \sigma^2 \rho \lambda S)(T-t)} \mathcal{N}(d_2, t) \right]
\]

which can be written as a function \( c(t, X_t) \) of \( t \) and \( X_t \). Using (A.2), we obtain that:

\[
\xi_t = \frac{\sigma x X_t}{\sigma S_t S_t} \rho c_X(t, X_t)
\]

The expression for \( \tilde{n}^F_t \) follows.

A.2 Proof of Proposition 2

Let us now assume that \( \Phi = \Phi^* \) and \( A_t = c(t, X_t) \). Under \( \mathbb{Q} \), \( A \) evolves as:

\[
dA_t = rA_t \, dt + \rho \sigma_X X_t c_X(t, X_t) \, dz^{S,Q}_t + d\tilde{C}_t \tag{A.3}
\]

\( (e^{-rt} c(t, X_t))_t \) also follows a \( \mathbb{Q} \)-martingale, whence:

\[
dc(t, X_t) = rc(t, X_t) \, dt + c_X(t, X_t) \sigma_X X_t \, dz^{X,Q}_t \tag{A.4}
\]

where \( dz^{X,Q}_t = dz^X_t + \rho \lambda S dt \) defines a \( \mathbb{Q} \)-Brownian motion. One can rewrite it as:

\[
dz^{X,Q}_t = dz^X_t - \rho \, dz^S_t + \rho \, dz^{S,Q}_t = \sqrt{1 - \rho^2} \, dz^2_t + \rho \, dz^{S,Q}_t \tag{A.5}
\]

\(^{13}\) This means that the product \( L^H \tilde{S} \) follows a \( \mathbb{Q} \)-martingale.
Comparing (A.3) and (A.4) we obtain that:
\[ e^{-rt} dC_t = e^{-rt}c_x(t, X_t)\sigma_x X_t \sqrt{1 - \rho^2} dW_t \]
hence equation (1.7).

In what follows, we let \( \delta_t \) be the delta of the option at time \( t \) that is computed from the minimal entropy price, namely, \( \delta_t = c_x(t, X_t) \). We must now compute \( \mathbb{E}^P [\delta_t^2 X_t^2] \) for all \( t \) in \([0, T]\). If \( (\delta_t^2 X_t^2)_{t \geq 0} \) were a martingale, the task would be easy since we would have \( \mathbb{E}^P [\delta_t^2 X_t^2] = \delta_0^2 X_0^2 \). But \( (\delta_t^2 X_t^2)_{t \geq 0} \) is not a martingale. However, it can be shown that (Lacoste (1996)), for small \( \sigma_x^2 T \), the expectation \( \mathbb{E}^P [\delta_t^2 X_t^2] \) is approximated by the constant \( \delta_0^2 X_0^2 \). The formal proof follows from a straightforward translation of a similar derivation in Lacoste (1996), where it is shown that \( \mathbb{E}^P [Y_t X_t^2] \approx y_0 X_0^2 \), \( Y_t = c_{xx}(t, X_t) \), denoting the gamma of the option. Therefore, we finally get
\[ \sigma_x^2 (1 - \rho^2) \int_t^T e^{-2r(s-t)} \mathbb{E}^P [\delta_s^2 X_s^2] ds \approx \sigma_x^2 (1 - \rho^2) \delta_0^2 X_0^2 \frac{1 - e^{-2(T-t)}}{2r} \]
which concludes the proof.

A.3 Proof of Proposition 3
Comparing (1.5) and (1.8) shows that:
\[ d((A_t - V_t)e^{-rt}) = e^{-rt} dC_t \]
Integrating from 0 to \( T \), together with the initial condition \( A_0 = V_0 \), yields:
\[ \mathbb{E}^P [(A_T - V_T)^2] = \mathbb{E}^P \left[ \left( \int_0^T e^{r(t-s)} dC_s \right)^2 \right] \]

A.4 Proof of Proposition 4
We take \( V \) and \( X \) as the state variables in this problem: this means that the relevant information for the investor's decision at time \( t \) is embedded in \((V_t, X_t)\). Hence the value function is defined as:
\[ v(t, V_t, X_t) = \sup_{\{\phi_t\}_{t \leq T}} \mathbb{E}^P_t \left[ -e^{-\alpha(V_T - (X_T - K)^+)} \right] \]
The dynamic programming principle implies that \( v \) solves:
\[ v_t + \sup_{\phi_t} \left[ v_V [r V_t + \phi_t (\mu_S - r)] + X_t v_X \mu X + X_t v_{XX} \phi_t \sigma_S \rho \sigma_X \right. \]
\[ \left. + \frac{1}{2} v_{XX} \sigma^2 X_t^2 + \frac{1}{2} v_{VV} \sigma^2 \phi_t^2 \right] = 0 \]
(A.6)
The first-order condition with respect to \( \phi_t \) yields:
\[ \phi_t^{**} = \frac{v_V}{v_{VV}} \frac{\lambda_S}{\sigma_S} - \frac{\rho X_t v_{XX}}{v_{VV}} \frac{\sigma_X}{\sigma_Y} \]
(A.7)
The value function must be a solution to the Hamilton-Jacobi-Bellman partial differential equation, which is obtained by substituting (A.7) into (A.6):
\[ v_t + v_V \left[ r - \frac{v_V}{v_{VV}} \frac{\lambda_S}{\sigma_S} - \frac{X_t v_{XX}}{V_t v_{VV}} \lambda_S \rho \sigma_X \right] + X_t v_X \mu X \]
\[ + \frac{1}{2} v_{VV} \left( \frac{v_V}{v_{VV}} \right)^2 \frac{\lambda_S^2}{\sigma_S^2} + X_t \frac{v_{XX}}{v_{VV}} \left( \frac{v_X}{v_{VV}} \right)^2 \sigma^2 \rho^2 + 2X_t \frac{v_{VV} v_{XX}}{v_{VV}} \lambda_S \rho \sigma_X \]
\[ + \frac{1}{2} v_{XX} \sigma^2 X_t^2 - v_{XX} \left( \frac{v_V}{v_{VV}} X_t \sigma_X \lambda_S \rho + \frac{X_t}{v_{VV}} \frac{v_{XX}}{v_{VV}} \sigma^2 \rho^2 \right) = 0 \]
(A.8)

14 - The approximation actually also requires a small \( \lambda_X \) or \( \delta_t \), where \( \lambda_X \) is the market price of risk (\( \lambda_X = 0 \) if agents are risk-neutral).
A qualified guess for the value function is:

\[ w(t, V, X) = k(t) \exp \left[ -\alpha e^{(T-t)}(V - b(t, X)) \right] \]  

(A.9)

where \( k \) and \( b \) are differentiable functions satisfying the terminal conditions \( k(T) = -1 \) and \( b(T, X) = (X - K)^+ \). Taking the relevant partial derivatives of the function \( w \) defined by (A.9) and substituting them in (A.8), we see that we will have \( v = w \) as long as \( k \) and \( b \) solve the following ordinary differential equations:

\[ \frac{k'}{k} = \frac{\lambda^2}{2} \]  

(A.10)

\[ 0 = \left[ \frac{1}{2} \sigma_X^2 X^2 b_{XX} - rb + rXb_X + b_t \right] + Xb_X \left( \mu_X - r - \sigma_X \rho \lambda_S \right) + \frac{1}{2} \sigma_X^2 e^ {r(T-t)} b_X^2 (1 - \rho^2) \sigma_S^2 \]  

(A.11)

The optimal strategy is obtained by substituting partial derivatives of \( v \) into (A.7):

\[ \phi^{*, \alpha}_t = \frac{e^{-r(T-t)} \lambda_S}{\alpha} + \rho X_t \frac{\sigma_X}{\sigma_S} b_X \]  

(A.12)

The optimal strategy when no option is written is obtained by letting \( b \) be equal to zero in \( \phi^{*, \alpha}_t \). The new value function in this context is denoted \( v_0(t, V_t) \). (A.9) implies that:

\[ v_0(t, V_t) = v(t, V_t + b(t, X_t), X_t) \]

which shows that \( b(t, X_t) \) is the indifference selling price of the option in the sense of Hodges and Neuberger (1989). The optimal strategy in the absence of the option coincides with the solution to Merton’s problem (Merton, 1971):

\[ \phi^{*, \alpha}_t = \frac{e^{-r(T-t)} \lambda_S}{\alpha} \]  

(A.13)

Given the definition of the indifference hedging strategy, we get that:

\[ \phi^{*, \alpha}_t = \frac{\sigma_X X_t}{\sigma_S} \rho b_X(t, X_t) \]

A.5 Proof of Propositions 5 and 6

To obtain (1.11), we follow Musiela and Zariphopoulou (2004) and we linearise equation (A.11) by letting \( d(t, X) = e^{\alpha(1-\rho^2)e^{r(T-t)}b(t, X)} \). (A.11) is thus equivalent to:

\[ 0 = \frac{1}{2} \sigma_X^2 X^2 d_{XX} + d_t + Xd_x \left( \mu_X - \sigma_X \rho \lambda_S \right) \]

with the terminal condition \( d(T, X) = e^{\alpha(1-\rho^2)(X-K)^+} \). Feynman-Kac’s theorem then shows that \( d(t, X) \) admits the following probabilistic representations:

\[ d(t, X_t) = \mathbb{E}^P_t \left[ e^{\alpha(1-\rho^2)e^{r(T-t)}(X_t - K)^+} \right] \]

\[ d(t, X_t) = \mathbb{E}^Q_t \left[ e^{\alpha(1-\rho^2)e^{r(T-t)}(X_t - K)^+} \right] \]

Now consider the case \( \alpha = 0 \). The ODE satisfied by \( b \), (A.11), becomes:

\[ 0 = \frac{1}{2} \sigma_X^2 X^2 b_{XX} - rb + b_t + Xb_X \left( \mu_X - \sigma_X \rho \lambda_S \right) \]
with the terminal condition \( b(T, X) = (X_T - K)^+ \). But the fact that \( (e^{-rt} c(t, X_t))_t \) is a \( Q \)-martingale implies that \( c \) solves:
\[
rc = c_t + c_X(\mu_X - \rho \sigma_X \lambda_S) + \frac{1}{2} \sigma_X^2 X^2 c_{XX}
\]
which is the same PDE as \( b \), with the same terminal condition \( c(T, X) = (X_T - K)^+ \). Hence \( b \) (resp., \( b_X \)) is equal to \( c \) (resp., \( c_X \)).

A.6 Proof of Proposition 7
The results in this proposition are easy consequences of proposition 1, having recognised that \( S = X \) implies that \( \rho = 1, \sigma_X = \sigma_S \) and \( \rho \lambda_S = \frac{\mu_X - \rho}{\sigma_X} \).

A.7 Proof of Proposition 8
We let \( \Delta t = t_{i+1} - t_i \). Over the interval \([t_i, t_{i+1}]\), the changes in values of the self-financing and of the non-self-financing strategies are:
\[
V_{t_{i+1}}^{\text{disc}} = n_i^S [\Delta S_{t_{i+1}} - S_{t_i} (e^{\rho \Delta t} - 1)] + (e^{\rho \Delta t} - 1) V_{t_i}^{\text{disc}}
\]
\[
\Delta A_{t_{i+1}}^{\text{disc}} = n_i^S [\Delta S_{t_{i+1}} - S_{t_i} (e^{\rho \Delta t} - 1)] + (e^{\rho \Delta t} - 1) A_{t_i}^{\text{disc}} + \Delta C_{t_{i+1}}
\]
Hence:
\[
\Delta [e^{-rt} (V_t^{\text{disc}} - A_t^{\text{disc}})] = -e^{-rt_{i+1}} \Delta C_{t_{i+1}}
\]
Summing up increments from \( i = 1 \) to \( i = n \), we obtain that:
\[
e^{-rT} (V_T^{\text{disc}} - (X_T - K)^+) - (V_0^{\text{disc}} - A_0^{\text{disc}}) = \sum_{i=1}^{n} e^{-rt_i} \Delta C_{t_i}
\]
This concludes the proof, since \( A_0^{\text{disc}} = V_0^{\text{disc}} \).

A.8 Proof of Proposition 9
As shown in Leland (1985), the change in value of the portfolio is:
\[
\Delta A_{t_{i+1}}^{\text{disc}} = A_{t_{i+1}}^{\text{disc}} - A_{t_i}^{\text{disc}} = \Delta BS_{t_i} + \frac{1}{2} \sqrt{\lambda_t} Y_t X_{t_i}^2 \left[ \frac{\sigma_X^2 \Delta t - \left( \frac{\Delta X_{t_{i+1}}}{X_{t_i}} \right)^2}{\sigma_X} \right] + o(\Delta t)
\]
where \( o(\Delta t) \) denotes the product of \( \Delta t \) and a quantity shrinking to zero as the trading period goes to zero. The incremental cost of the discrete strategy at time \( t_{i+1} \) is then \( \Delta C_{t_{i+1}} = \Delta BS_{t_{i+1}} - \Delta A_{t_{i+1}}^{\text{disc}} \) and the residual risk at time 0 is:
\[
R_0^{\text{disc}} = \mathbb{E}^P \left[ \sum_{i=0}^{N-1} (\Delta C_{t_{i+1}})^2 \right] + \mathbb{E}^P \left[ \sum_{i<j} (\Delta C_{t_{i+1}})(\Delta C_{t_{j+1}}) \right]
\]
Applying the law of iterated conditional expectations, we get that:
\[
R_0^{\text{disc}} = \mathbb{E}^P \left[ \sum_{i=0}^{N-1} \mathbb{E}_{t_i}^P \left[ (\Delta C_{t_{i+1}})^2 \right] \right] + \mathbb{E}^P \left[ \sum_{i<j} \mathbb{E}_{t_i}^P \left[ (\Delta C_{t_{i+1}})(\Delta C_{t_{j+1}}) \right] \right]
\]
From now on, we ignore the \( o(\Delta t) \) term in (A.14). Then we have \( \mathbb{E}_{t_{i+1}} \left[ \Delta C_{t_{i+1}} \right] = 0 \), and \( \Delta C_{t_{i+1}} \) is \( \mathcal{F}_{t_{i+1}} \)-measurable, so \( \mathbb{E}_{t_{i+1}} \left[ (\Delta C_{t_{i+1}})(\Delta C_{t_{j+1}}) \right] = 0 \). Hence the aggregate risk of the discrete strategy:
\[
R_0^{\text{disc}} \approx \mathbb{E}^P \left[ \sum_{i=0}^{N-1} \mathbb{E}_{t_i}^P \left[ (\Delta C_{t_{i+1}})^2 \right] \right]
\]
Using (A.14), we get that:

\[
R_0^{\text{disc}} \approx \frac{1}{4} \mathbb{E}^p \left[ \left( \sum_{i=0}^{N-1} \gamma_i^2 X_{t_i}^4 \right) \mathbb{E}^p \left( \left( \sigma_X^2 \Delta t - \left( \frac{\Delta X_{t_{i+1}}}{X_{t_i}} \right)^2 \right) \right)^2 \right]^{-\frac{1}{2}}
\]

As seen from date \( t_i \), \( \Delta X_{t_{i+1}}/X_{t_i} \) is normally distributed with mean \( \mu_X \Delta t \) and variance \( \sigma_X^2 \Delta t \), so:

\[
\mathbb{E}^p \left[ \left( \sigma_X^2 \Delta t - \left( \frac{\Delta X_{t_{i+1}}}{X_{t_i}} \right)^2 \right) \right] \approx 2\sigma_X^4 (\Delta t)^2 + 4\mu_X^2 \sigma_X^2 (\Delta t)^3 + \mu_X^4 (\Delta t)^4
\]

Ignoring terms in \((\Delta t)^3\) and \((\Delta t)^4\), we obtain that:

\[
R_0^{\text{disc}} \approx \frac{1}{2} \sigma_X^4 (\Delta t)^2 \sum_{i=0}^{N-1} \mathbb{E} \left[ \gamma_i^2 X_i^4 \right]
\]

Approximating each term in the sum with \( \gamma_0^2 X_0^2 \), we get:

\[
R_0^{\text{disc}} \approx \frac{1}{2} \sigma_X^4 (\Delta t)^2 N \gamma_0^2 X_0^4 = \frac{1}{2} \sigma_X^4 T \Delta t \gamma_0^2 X_0^4
\]

In the Black-Scholes model, we have that \( \gamma_0 = \frac{n(d_{1.10}^{\text{emp}})}{\sigma \sqrt{T}} \), which concludes the proof.
This table displays the base-case parameters. $\sigma_X$ and $\lambda_X$ (resp. $\sigma_S$ and $\lambda_S$) denote the volatility and the Sharpe ratio of the non-traded underlying asset (resp. of the traded substitute). $r$ is the risk-free rate. The option is assumed to be at-the-money at the initial date, and the two assets are assumed to start from the same initial value, $X_0 = S_0 = 1$.

### Figure 1: Risk of the locally-risk-minimising strategy

This figure plots the risk of the locally risk-minimizing strategy. $\sigma_X$ denotes the volatility of the non-traded underlying of the option, and the volatility of the substitute ($\sigma_S$) is taken equal to $\sigma_X$. Unless otherwise indicated, parameters are fixed at their base-case values (see table 1).

### Figure 2: Locally-risk-minimising strategy vs heuristic strategy – Increase in risk

This figure plots the ratio of the risk of the heuristic strategy based on the Black-Scholes delta computed with the parameters of the underlying asset, over the risk of the locally-risk-minimising strategy. Both strategies use the imperfect substitute. $\sigma_X$ denotes the volatility of the non-traded underlying of the option, and the volatility of the substitute ($\sigma_S$) is taken equal to $\sigma_X$. Other parameters are fixed at their base-case values (see table 1).

---

**Table 1: Base-case parameters**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
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<tr>
<td>$\sigma_X$</td>
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<tr>
<td>$\sigma_S$</td>
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</tr>
<tr>
<td>$\lambda_X$</td>
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<tr>
<td>$\lambda_S$</td>
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<tr>
<td>$r$</td>
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<tr>
<td>$S_0/X_0$</td>
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<tr>
<td>$X_0$</td>
<td>1</td>
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</tbody>
</table>
Figure 3: Locally-risk-minimising strategy vs discrete-time strategy

(a) Risk of discrete-time strategy.

(b) Breakeven correlations as functions of the discrete time interval.

These figures plot the risk associated with the discrete strategy and the break-even correlations for various trading frequencies and various maturities of the option. The break-even correlation is the minimum correlation needed for the locally-risk-minimising strategy based on the imperfect substitute to perform better than a discrete-time strategy based on the true underlying. $\sigma_X$ denotes the volatility of the non-traded underlying, and the volatility of the substitute ($\sigma_S$) is taken equal to $\sigma_X$. Other parameters are fixed at their base-case values (see table 1).
Figure 4: Heuristic strategy in the presence of transaction costs
(a) First moment of residual risk.

These figures plot the first two moments of the residual risk at time 0 normalised by the Black-Scholes price of the option. This ratio is plotted as a function of trading frequency for different maturities of the option and different levels of the proportional transaction costs $k$. Unless otherwise indicated, parameters are fixed at their base-case values (see table 1).
References


