Mass Customisation versus Mass Production in Retirement Investment Management: Addressing a "Tough Engineering Problem"

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Executive Summary
Existing financial products marketed as “retirement investment solutions” do not meet the needs of future retirees, which involve securing their essential goals expressed in terms of minimum levels of replacement income (focus on safety), while generating a relatively high probability of achieving their aspirational goals expressed in terms of target levels of replacement income (focus on performance). Meaningful solutions should therefore combine safety and performance to meet this dual objective.

With defined-contribution (DC) pension schemes increasingly replacing defined-benefit (DB) plans in most developed countries, individual investors are increasingly left with the necessity to decide for themselves how to finance their consumption needs in retirement. Since many of them do not have the financial background to make educated investment decisions, this profound evolution leaves the investment management industry with the responsibility of providing them with suitable retirement solutions.

Existing retirement products, however, fall short of providing satisfactory solutions to the problems faced by individuals when approaching investment saving decisions. Among currently available products, target-date funds are often used as a default option in DC plans, but they offer a sole focus on investment horizon without any protection of investors’ minimum retirement needs. In particular, these products are not engineered to deliver replacement income in retirement, and do not achieve a proper hedging of the main risks related to the retirement investing decisions, namely investment risk, interest rate risk, inflation risk and longevity risk. Another important restriction is that most existing target-date funds do not allow for revisions of the asset allocation as a function of changes in market conditions. This is entirely inconsistent with academic prescriptions, and also common sense, which both suggest that the optimal strategy should also display an element of dependence on the state of the economy.

In contrast, annuities and variable annuities, which are marketed by insurance companies, deliver a predetermined income in retirement, possibly including a cost of living adjustment, either for life or for a fixed period. While they are, in principle, the safe asset with respect to the replacement income goal, their cost-inefficiency and lack of reversibility makes them ill-suited investment vehicles, especially in the accumulation phase.

Currently available investment options hardly provide a satisfying answer to the retirement investment challenge, and most individuals are left with an unsatisfying choice between on the one hand safe strategies with very limited upside potential, which will not allow them to generate the kind of target replacement income they need in retirement, and on the other hand risky strategies offering no security with respect to minimum levels of replacement income. This stands in contrast with a well-designed retirement solution that would allow individual investors to secure the level of replacement income in retirement needed to meet their essential consumption goals, while generating a relatively high probability of them achieving their aspirational consumption goals, with possible additional goals including...
healthcare, old age care and/or bequests. This recognition is leading to a new investment paradigm, which has been labelled goal-based investing (GBI) in individual money management, where investors’ problems can be fully characterised in terms of their lifetime meaningful goals, just as liability-driven investing (LDI) has become the relevant paradigm in institutional money management, where investors’ problems are broadly summarised in terms of their liabilities. The GBI framework puts investor’s goals and constraints, not the characteristics of investment products, at the heart of the investment decision process. This approach explicitly recognises that true retirement investment solutions should meet the needs of future retirees, which are to generate enough replacement income to finance their expenses in retirement.

A theoretical answer to the retirement investment problem is a dynamic goal-based investing strategy that maximises the probability of reaching a target level of income (aspirational goal) while securing a minimum (essential goal). In practice, this strategy is hardly implementable, especially if full customisation is not possible.

Financial theory and stochastic calculus can serve as useful guides towards the design of investment strategies that reconcile safety and performance with respect to replacement income goals. The first step is to measure the price of one dollar of replacement income, by using the appropriate discount rates, a coefficient of indexation where applicable, and a mortality table. The result is the actuarially fair price of an annuity that delivers a unit replacement income, a price that can vary because of interest rate changes and revisions in mortality probabilities. The income that can be financed with a given capital is obtained by dividing the capital by the annuity price, which can be used to distinguish between affordable and non-affordable goals. In the affordability computation, one can decide to rely only on current wealth or on additional future expected contributions as well.

The retirement investment problem can be mathematically expressed as follows: given a contribution schedule, a target level of income unaffordable with the available resources and an affordable minimum level, maximise the probability of reaching the target by the retirement date, while securing the minimum. The target income level is called the aspirational goal, and the minimum represents the essential goal. The solution is reminiscent of a binary option, but one that is written on terms of replacement income, with only two possible outcomes at the retirement date—the target and the minimum levels of income. In a Black-Scholes setting with constant risk and return parameters, it is possible to replicate this payoff by dynamically investing in the maximum Sharpe ratio portfolio and the goal-hedging portfolio (GHP) defined as the annuity-replicating portfolio. In addition, when periodic contributions take place, the dynamic replication strategy involves a short position in the accumulation bond, which is the coupon-paying bond with cash flows equal to the expected contributions.

While theoretically optimal, the probability-maximising strategy is impossible to implement in practice because it requires
continuous trading, unreasonably high leverage levels and the use of unobservable and difficult to estimate parameters such as expected returns on risky assets. Moreover, it entirely depends on an investor’s subjective characteristics, notably the aspirational goal level, which can greatly vary from one individual to the other. This is a serious obstacle in retail money management, where full customisation is barely possible.

To be consistent with mass distribution constraints, a strategy should be scalable, meaning that it should be able to secure the essential goals and have a high probability of reaching the aspirational goals for a population of investors. This can be achieved by combining elementary strategies referred to as “(1,0)” and “(1,1)” building blocks and by using a suitable allocation to a performance-seeking portfolio and the goal-hedging portfolio within each strategy.

Investors can differ in gender, retirement age, current age, contribution level, essential and aspirational goals, and also in the date at which they start to accumulate for retirement. The “tough engineering problem”, as Merton (2003) puts it, is to design a limited set of mass-customised investment solutions that can adequately address the needs of a heterogeneous population.

To simplify the problem, one may first focus on providing replacement income for a fixed period of time, say 20 years, in retirement and possibly consider using late life annuities for getting protection against tail longevity risk. In this case, longevity risk, which is gender dependent, does not impact the design of the retirement solution. With respect to age and retirement date, one might assume a few different fixed retirement ages (say 65Y and 70Y) and group individuals in age-based clusters (say 35Y, 40Y, 45Y, 50Y, 55Y, 60Y), as done in the target date fund industry.

Cross-sectional dispersions in contribution levels pose a priori a more serious scalability problem. Indeed if one considers a general contribution scheme defined as \((C_0, C)\), where \(C_0\) is the initial contribution and \(C\) is the level of future contributions, investors may widely differ in terms of the levels of and ratio between initial and future contributions.

Fortunately, it turns out that this large variety of investors can be addressed with two elementary dynamic GBI strategies, corresponding respectively to \((C_0 = 1, C = 0)\) and \((C_0 = 1, C = 1)\). Rather than creating a \((C_0, C)\) fund for each and every possible value of \(C_0\) and \(C\), each given investor will instead be offered a buy-and-hold allocation to each of these two elementary funds, whereby they would initially invest \(C_0 - C\) in the \((1, 0)\) elementary strategy, and the remaining amount \(C\) in the \((1, 1)\) strategy. Every year, the new contribution \(C\) will be allocated to the \((1, 0)\) strategy while additional unscheduled contributions, if any, would be allocated to the \((1, 0)\) strategy. An even more parsimonious approach would consist in investing all scheduled or unscheduled contributions in the \((1, 0)\) strategy. This allows us one to avoid the use of \((1, 1)\) strategies that may raise implementation concerns related to their embedded short position in the accumulation bond.

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Then, a common essential goal is fixed in the form of a percentage, say 80%, of the initially affordable income to be secured by the retirement date. In other words, if the expected contribution schedule allows an annual income of $10,000 to be financed given the interest rate and longevity conditions prevailing at the beginning of accumulation, the essential goal is to have at least $8,000 in retirement. The remaining 20% of purchasing power in terms of annuities correspond to a risk budget that is put at risk in order to get a chance to eventually receive more than $10,000 per year. By adopting the risk-free strategy of investing only in the GHP, the affordable replacement income would stay constant at $10,000 per year.

Each one of the two elementary strategies shall secure the essential goal of any investor, regardless of the entry date in the fund. This is done through a dynamic allocation to three building blocks: (1) a PSP, which is intended to have a high Sharpe ratio; (2) the GHP, which tracks the present value of the income stream; and (3) the accumulation bond, to make up for the presence of regular contributions. The dollar allocation to the PSP is a function of the risk budget, defined as the difference between the current fund value and a floor, which shrinks to zero as the risk budget vanishes. This investment rule is qualitatively similar to that of the probability-maximising strategy, except that it involves no unobservable parameter, no leverage, and can be implemented with discrete, say quarterly, rebalancing. Broadly speaking, it can be regarded as an extension of constant proportion portfolio insurance and dynamic core-satellite techniques to the retirement investing context. The allocation to the performance building block equals the risk budget times a multiplier, which can be a constant greater than one or a time-varying quantity, and aims to increase the access to the upside potential.

The variation of the aspirational income level across individuals is addressed by implementing a stop-gain mechanism. The investment policy of each fund is independent from any aspirational goal, and each investor can exit it and transfer their assets to the GHP as soon their targets are attained. As a result, we have a set of funds scalable with respect to the entry point, the contribution level and the aspirational goal, which implies a significant reduction in the number of funds to maintain.

New funds should be launched to address the needs of new cohorts or to replace existing funds whose risk budget has been exhausted. Numerical simulations show that the parsimonious creation of new funds has a positive effect on the upside potential for new investors. Moreover, only goal-based investing strategies can reliably secure essential goals, unlike balanced or target-date funds, and they have attractive probabilities of reaching aspirational goals.

The first reason why a new fund has to be created is because new cohorts have retirement dates that exceed the maturities of existing funds. The timing of these fund launches can be anticipated with certainty. A second reason is because an existing (1,0) or (1,1) fund has become sterilised or quasi-sterilised following a strong underperformance of the PSP with respect to the GHP, and the upside potential is
too low to make the fund attractive to new investors or even to existing investors wanting to add contributions to their plan. The risk of sterilisation is inherent to portfolio insurance, and the negative consequence is an increase in the number of funds to manage. Fortunately, it can be mitigated by implementing a “first line of defence” mechanism through a systematic reset of the floor. If the fraction of affordable income to protect is for example 80%, we fix it to a higher level at the fund inception, say 85%, and decrease it down to 80% when the risk budget is too low. Two improvements are expected with respect to a situation without resets, namely increase the upside potential for new investors, and delay sterilisation time. These effects can be verified by performing Monte-Carlo simulations of the performance of funds. We generate 10,000 random scenarios for the PSP and interest rates, as well as corresponding returns for the GHP and the (1,0) and (1,1) elementary GBI strategies. Exhibit 1 was obtained from such simulations, and it shows that resets do slightly improve the probabilities of reaching aspirational goals for investors who arrive after inception. It is also seen that they imply a decrease in the probability of having to launch new funds to replace existing ones.

Goal-based retirement strategies clearly dominate other approaches based on balanced or deterministic target-date funds in terms of the probabilities of securing the essential goal, as illustrated in Exhibit 2. The numbers in the column “Optimal” are unattainable upper bounds, since they are obtained with the theoretical probability-maximising strategies. Strategy S1 uses a customised \((C_0, C)\) fund, while S2 invests all contributions in a highly scalable \((1, 0)\) fund. The comparison between S1 and S2 shows the opportunity cost arising from the unavailability of customised \((C_0, C)\) funds for non-zero \(C\). The next strategies displayed in Exhibit 2 are heuristic policies: H1 is fully invested in the PSP; H2 is an equally-weighted portfolio of the PSP and a generic bond index with no attempt at matching the duration of the GHP; and H3 implements a deterministic glide

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**Exhibit 1:** Effect of resets on success probabilities for new investors (left) and on probabilities of launching at least \(N\) funds within the accumulation period (right).

Note: This figure relates to hypothetical \((1,0)\) funds launched in January 2016 for retirement in 2046. “NR” and “R” refer respectively to the cases without and with resets. Along the curves “Initial NR” and “Initial R”, new investors invest in the initial fund, and curve “Latest R” shows the probabilities for those who invest in the most recently launched fund.
path from the PSP to the bond index, which proxies for the investment policy of deterministic target-date funds. (The allocation to the PSP decreases from 90% to 30% in these simulations.)

Only strategies S1 and S2 reliably secure the essential goal, with the heuristic policies having significant shortfall probabilities, of about 15% at least. This is a concern because failure to reach the essential goal may leave the individual short of resources to finance minimal consumption expenses in retirement. More importantly, the 100% probability for S1 is not model- or parameter-dependent, unlike other probabilities. Indeed, S1 uses the truly safe asset (the annuity-replicating portfolio), which perfectly replicates the present value of the essential goal regardless of other parameter values. As far as aspirational goals are concerned, strategy S1 delivers attractive probabilities, of about 70% for aspirational goals up to 130% of the initially affordable income. When the investor has scheduled contributions, the unavailability of the (1,1) fund implies a non-negligible loss in probability, but the loss shown here can be regarded as an upper bound given that all contributions are the same size. In practice, the initial one is usually larger, so $C/C_0$ is less than 1, and the opportunity cost of having only the (1,0) fund is lower.

As a conclusion, dynamic goal-based investing principles can be used to design a parsimonious set of retirement investment strategies which meet the needs of individual investors preparing for

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<th>Probabilities of reaching essential and aspirational goals (in %).</th>
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<td>Aspirational (%)</td>
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<td>Essential</td>
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Note: This table relates to the case of a hypothetical investor starting to accumulate in January 2016 and planning to retire in January 2046. $C/C_0$ is the ratio of the periodic annual contribution (C) to the initial one ($C_0$). Optimal probabilities are by construction independent from $C/C_0$. The multiplier in strategies S1 and S2 is taken equal to 3.
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Executive Summary

retirement in that they secure an essential level of replacement income and also have good probabilities of generating much more replacement income than what they would have obtained by investing in annuities, and this is possible in a cost-efficient and reversible format. These goal-based investing principles can be applied to other goals relevant to a large class of investors and households, such as financing their children’s further education.
Introduction

Triggers by the introduction of ever stricter accounting and prudential pension fund regulations, a massive shift from defined-benefit to defined-contribution pension schemes is taking place across the world. As a result of this trend, individuals are becoming increasingly responsible for making investment decisions related to their retirement financing needs, investment decisions that they are not equipped to deal with given the low levels of financial literacy within the general population. In the context of such a massive shift of retirement risks to individuals, the investment management industry is facing an ever greater responsibility in terms of the need to provide households with suitable retirement solutions.

Unfortunately, currently available investment products manufactured by most asset managers or insurance companies hardly provide a satisfying answer to investors’ and households’ replacement income needs in retirement. On the asset management side, target date funds, which are often used as a default option in retirement plans, generally focus on reducing the uncertainty over capital value near the retirement date, regardless of the decumulation objectives in terms of replacement income in retirement. As a result, they typically offer no protection to investors against unexpected changes in interest rates, inflation and longevity, which are the risk factors that affect the present value of an income stream. Alongside their asset-only focus, and as noted by Bodie, Detemple, and Rindisbacher (2009), such investment policies are only rough approximations for strategies that maximise expected utility over the life cycle, and Cocco, Gomes, and Maenhout (2005) show that the utility cost borne by individuals who use them can be substantial. Turning to insurance products, inflation-linked deferred annuities are effectively designed to deliver a lifetime replacement income that guarantees a fixed purchasing power in terms of consumption goods and the use of these products can be rationalised in optimal portfolio choice models. Yaari (1965) shows that an investor who has an uncertain lifetime and no utility for bequest should fully annuitise his/her wealth, provided annuities are fairly priced in the actuarial sense. Cocco and Gomes (2012) find that agents with uncertain longevity can enjoy significant utility gains by investing in longevity bonds, which insures them against the risk of living longer than expected. However, the observed demand for annuities remains low, a fact referred to as the “annuity puzzle”. Brown (2001) recognises that a life-cycle portfolio choice model only partially explains the empirical choices of individuals with respect to annuitisation. Pashchenko (2013) surveys various reasons that explain the low level of annuitisation, such as the existence of “pre-annuitised” wealth (Social Security and defined-benefit plan benefits), adverse selection ruling out groups with higher mortality, and frictions such as minimum investment, irreversibility, etc. Other reasons include the perceived cost-inefficiency of annuities, which are not sold at actuarially fair prices (Friedman and Warshawsky (1988, 1990)), the fact that they do not contribute to bequest objectives, and the fact that annuitised wealth cannot be recovered in the form of capital even if the beneficiary experiences
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a severe health problem that would generate large expenses (Peijnenburg, Nijman, and Werker, 2015).1

Beyond their respective intrinsic limitations, target date funds or annuities suffer from one fatal flaw, namely they are off-the-shelf investment products when investors need dedicated investment solutions tailored to address their specific needs and constraints. Mass production (in terms of investment products) happened a long time ago in investment management, through the introduction of mutual funds and, more recently, exchange-traded funds. Now, the effective challenge is mass customisation (as in customised investment solutions), which by definition is a manufacturing and distribution technique that combines the flexibility and personalisation of "custom-made" solutions with the low unit costs associated with mass production. In other words, the challenge is indeed to find a way to provide a large number of individual investors with meaningful dedicated investment solutions.

That mass customisation is the key challenge that our industry is facing has long been recognised, but only recently have we developed the actual capacity to provide individuals with such dedicated investment solutions. This point was made very explicitly by Merton (2003): “It is, of course, not new to say that optimal investment policy should not be 'one size fits all'. In current practice, however, there is much more uniformity in advice than is necessary with existing financial thinking and technology. That is, investment managers and advisors have a much richer set of tools available to them than they traditionally use for clients. (...) I see this issue as a tough engineering problem, not one of new science. We know how to approach it in principle (...) but actually doing it is the challenge.”

Paraphrasing Robert Merton, we emphasise that designing meaningful retirement solutions does not indeed require a new science. In a nutshell, the challenge to finance substantial levels of consumption in retirement with limited dollar budgets (contributions) as well as limited risk budgets can be addressed through a disciplined use of the three forms of risk management, namely risk hedging (for an efficient control of the risk factors in investors' retirement goals), diversification (for an efficient harvesting of risk premia) and insurance (for securing essential goals while generating attractive probabilities of attaining aspirational goals). The first contribution of this paper is precisely to analyse how the retirement investing problem, which generically involves minimum and target levels of replacement income in retirement (corresponding to retirees' "essential" and "aspirational" goals in the terminology of Wang et al. (2011)), can be formally framed within the context of dynamic portfolio choice theory. The literature has provided solutions to the problem of maximising the probability of reaching a target wealth level while securing a minimum wealth level (Browne (1999), Föllmer and Leukert (1999)). We first extend these results to a liability-driven context where essential and aspirational goals are expressed in terms of replacement income, as opposed to wealth, and also generalise these results to account for the presence of a stochastic opportunity

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1 - Variable annuities, which are annuity products that offer participation to the upside of equity markets, suffer from similar flaws, namely, unavailability early on in the accumulation phase, cost-inefficiency due to prohibitive cost of capital for insurers offering formal guarantees, as well as a lack of transparency and lack of flexibility, which leaves investors with no exit strategy, unless at the cost of high surrender charges.
set as well as intermediate contributions. The optimal strategy we derive is attractive in the sense of maximising by construction the probability of reaching the target while securing the minimum level of essential income, but this digital option payoff cannot be replicated in a realistic setting with discrete trading and leverage constraints. As an alternative, we analyse the mathematical conditions that a strategy should satisfy in order to meet the goals of future retirees, and we show that they can be fulfilled with a strategy that combines the same building blocks as the optimal one, but with a simplified and implementable allocation rule. The elementary building blocks are a performance-seeking portfolio (PSP), defined as a well-diversified portfolio of rewarded factor exposures, and a goal-hedging portfolio (GHP), which replicates a stream of inflation-linked cash-flows in retirement. In the presence of scheduled contributions, a third building block is involved, the role of which is to offset the implicit long interest rate exposure held through the contribution stream. The allocation to these blocks is a function of the risk budget, defined as the difference between the current asset value and a floor, which represents the value of the GHP that finances the essential goal. This investment rule is somewhat reminiscent to portfolio insurance strategies extended here to the case of a stochastic benchmark (see examples in Teplá (2001) and Martellini and Milhau (2012)).

The second and main contribution of this paper is to show that financial engineering can be used to address the “tough engineering problems” posed by the scalability requirements. Indeed, it is hardly feasible to launch a customised dynamic allocation strategy for each investor, and the challenge is to address the needs of a large number of investors through a limited number of funds. Parsimony is a priori difficult to achieve since individuals can differ with respect to multiple dimensions, including gender, current age, retirement age and date, contribution levels, as well as minimum and target replacement income needs, and not all of them start to accumulate on the same date. To address these questions, we first show that scalability with respect to the contribution level can be achieved by having individuals invest in two elementary funds: a so-called (1,0) fund designed for investors who make an initial contribution of $1 and have no scheduled contribution thereafter, and a so-called (1,1) fund intended for investors who plan to contribute $1 every year in accumulation. More generally, we consider (1, c) funds that are suitable for investors for whom the ratio of the annual regular contribution to the initial contribution is c. As far as the aspirational goal is concerned, the funds are perfectly scalable since the strategies do not involve this parameter. Probability-maximising strategies require that investors liquidate their portfolio and secure the aspirational goal by investing in the goal-hedging portfolio as soon as the target level of replacement income is reached. We suggest that this decision be left to investors and not embedded in the fund investment policy, thereby guaranteeing that the strategy is independent from any aspirational level and can thus accommodate the needs of a variety of investors with different replacement income targets. Another challenge relates to the dependency upon
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the entry point, which implies that even similar investors who would enter the investment solution at different points in time would need to follow two different strategies, thus raising again concerns in terms of scalability in implementation. We show that this issue can be addressed by introducing a “relative maximum drawdown floor” in each fund. This floor is a fraction, say 80%, of the maximum of the fund value expressed in the replacement income numeraire, thus ensuring that all investors have the same fraction of their initially affordable replacement income secured, regardless of their entry date. A potential limitation to scalability is that as the relative maximum drawdown floor increases, the fund allocation becomes more conservative, so that investors who arrive several years after the launch have lower probabilities of reaching a given aspirational level, even after taking into account the reduction in their investment horizon. We argue that an effective way to alleviate this concern is to reset the floor when it exceeds too large a fraction, say 95%, of the current fund value. On the other hand, resets are limited in size because of the need to protect the essential goals of existing investors. When resets are no longer possible, the fund becomes quasi-sterilised, and it has to be replaced by a new one in order to offer upside potential to new investors. Our results indicate that resets allow the time for creating a new fund to be significantly increased by delaying the quasi-sterilisation time. This result is practically important because it implies that the number of funds to manage is lower if resets are performed when needed. By integrating all of these ingredients, the needs of a large population of investors can be addressed using a limited number of funds. For example, if we assume the same retirement age for all individuals, we would only need 5 (1,0) funds, corresponding to investors in 5 different age groups, as well as 5 (1,1) funds, for a maximum total of 10 funds.

The rest of the paper is organised as follows. In Section 2, we define the goals in retirement planning and we introduce affordability criteria. In Section 3, we derive the strategy that maximises the probability of reaching a non-affordable target replacement income level while securing a minimum replacement income level. In Section 4, we establish the mathematical properties that a mutual fund must have in order to meet the objectives of a cohort of individuals. Section 5 describes investment strategies that satisfy these conditions and defines the elementary (1,0) and the (1, c) funds. In Section 6, we discuss more practical questions related to the implementation of the funds. In Section 7, we numerically evaluate the opportunity cost of mass-customised retirement solutions with respect to their fully customised counterparts and we show how reset rules and a limited number of new fund launches help investors reach their aspirational retirement goals. Section 8 concludes.
Introduction
2. Goals in Retirement Investing
2. Goals in Retirement Investing

The main concern for future retirees is to generate replacement income in retirement. Achieved levels of replacement income are uncertain, because they depend on market risks (which affect invested wealth levels), as well as on longevity, interest rate and inflation risks (which affect the price of lifetime replacement income). In this section, we introduce formal definitions for the notions of affordable replacement income and discuss priority rules across goals.

2.1 Definition of Goals

We consider an investor who starts to accumulate money for retirement at date 0 and lives at most until date $T_{\text{max}}$, so the time span of the model is $[0, T_{\text{max}}]$. Included in this range is the accumulation period $[0, T]$, $T$ being the retirement date. The time unit is the year, and $T$ and $T_{\text{max}}$ are integer numbers of years. Uncertainty in the economy is represented by a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\mathbb{P}$ represents the investor’s beliefs, equipped with a filtration $(\mathcal{F}_t)_{t \in [0, T_{\text{max}}]}$: $\mathcal{F}_t$ is the information set available at date $t$. For brevity, we denote expectations conditional on $\mathcal{F}_t$ as $\mathbb{E}_t$. There exists a stochastic discount factor $(M_t)_{t \geq 0}$, such that prices multiplied by $M_t$ follow martingales.

A goal is defined as a level of replacement income, which can be expressed as an annual income. It can be measured in current or constant dollars, with the latter option being more meaningful for investors seeking to secure consumption levels over long horizons. Let $r_{\text{max},t}$ be the maximal replacement income that an investor can finance at some date $t \leq T$. A goal $r_i$ is said to be affordable at date $t$ if it satisfies $r_i \leq r_{\text{max},t}$.

2.2 Measuring Affordable Income

To secure a stream of replacement income, the investor can purchase the safe asset for a retirement goal on the retirement date, which is defined as an immediate annuity (with possible inflation-indexation or at least cost of living adjustment). This safe asset is defined as a contract paying cash flows for the investor’s lifetime. The elementary annuity is a “unit contract”, which pays one dollar (in constant or current terms) per unit of time (month or year) from the retirement date until investor’s death. Letting $\beta_t$ be its price at a date $t$ in the accumulation phase, we have:

$$\beta_t = \mathbb{E}_t \left[ \frac{M_T}{M_t} \beta_T \right].$$

This is the present value of the immediate annuity price. Alternatively, given the cost inefficiency and irreversibility of annuities, the investor can purchase a ladder bond portfolio paying off replacement income for a fixed period of time in retirement, say for the first 20 or 25 years (and possibly separately acquire a late life annuity to provide protection against longevity risk). For simplicity in this paper, regardless of whether longevity risk is provided or not, we refer to this safe asset from the perspective of a retirement goal as an annuity. The simplest way of computing affordable replacement income is to evaluate the purchasing power of retirement savings in terms of annuities. At date $t$, the maximum replacement income $r_{\text{max},t}$ that can be financed with the capital $W_t$ is:

$$r_{\text{max},t} = \frac{W_t}{\beta_t}. \quad (1)$$
2. Goals in Retirement Investing

Some investors may have scheduled contributions, that is they expect to bring a minimum amount of money every year until retirement. In this case, we need to recognise that investor’s wealth does not restrict to his/her financial value but also includes a claim on future contributions. Let \( \alpha_t \) be the price of a bond that pays an annual coupon of $1 from date \( t \) excluded to date \( T \) included, which we refer to as the accumulation bond. With an annual contribution \( y \), the total wealth is the sum of the financial capital \( W_t \) and the present value of the contribution stream, \( y\alpha_t \). The replacement income that can be financed with this total wealth is:

\[
ri_{\max,t} = \frac{W_t + y\alpha_t}{\beta_t}.
\]  

This quantity coincides with (1) when \( y = 0 \), so we take it as the definition of the affordable income in what follows.

The affordability condition of a goal \( ri \) is equivalent to:

\[
W_t \geq ri\beta_t - y\alpha_t.
\]

In words, this inequality means that the investor’s liquid wealth must cover the price of a long-short bond, the long leg of which is an annuity that finances the desired income and the short leg of which is a bond that pays $1 every year until the retirement date. Should the long-short bond price be negative, the desired income would be affordable from future contributions alone and no liquid wealth would be needed.

2.3 Hierarchical Classification of Goals

The probability of reaching a goal \( ri \) (probability of success) is defined as \( \mathbb{P}(ri_{\max,T} \geq ri) \). The goal is said to be secured if the probability that it will be affordable at date \( T \) is one:

\[
\mathbb{P}(ri_{\max,T} \geq ri) = 1. \tag{3}
\]

By absence of arbitrage opportunities, if a goal is secured, it must be affordable at date \( 0 \), so the range of goals that can be secured is \([0, ri_{\max,0}]\), where \( ri_{\max,0} = (W_0 + y\alpha_0)/\beta_0 \).

Following Deguest et al. (2015), we make a distinction between two types of goals:

- **Essential goals** are affordable goals that the investor wants to secure “almost surely” in the probabilistic sense (3). Moreover, the 100% probability should be robust to the assumptions on the dynamics of risk factors that impact the affordable income;
- **Aspirational goals** are non-affordable goals that the investor does not expect to secure, but wants to reach with the highest possible probability.

Formally, the robustness condition means that condition (3) should be satisfied for any element \( \mathbb{P}_\theta \) of a family of probability measures \( (\mathbb{P}_\theta)_{\theta \in \Theta} \): \( \Theta \) characterises the set of all possible distributions for the risk factors; it is not necessarily a finite-dimensional space (e.g. if it encompasses all continuous diffusion processes).
2. Goals in Retirement Investing

In what follows, we represent the essential goal as a fraction $\delta_{\text{ess}} < 1$ of the affordable income of date 0. This parameter corresponds to the aversion for loss: $1 - \delta_{\text{ess}}$ is the largest loss in replacement income that the investor is willing to accept in exchange for the possibility of reaching an aspirational goal. $\delta_{\text{ess}}$ is thus a psychological parameter that depends on how the investor values the trade-off between upside potential and downside risk.

Similarly, the aspirational goal is a level of replacement income equal to $\delta_{\text{asp}} ri_{\text{max},0}$ where $\delta_{\text{asp}} > 1$. In fact, there may not be a well-defined level for this goal in all cases: the investor may be willing to move beyond the initially affordable level, but without having a specific target in mind. Thus, even for a single investor, there is a range of possible aspirational goals, and the upside potential of a given strategy should be assessed by evaluating the probabilities of reaching various aspirational levels.

At each date, the investor can evaluate the distance from the essential or the aspirational goals by computing the funded ratio:

$$R_t = \frac{ri_{\text{max},t}}{ri_{\text{max},0}}$$

and by comparing it to the essential and aspirational levels $\delta_{\text{ess}}$ and $\delta_{\text{asp}}$. This definition is similar to that adopted by Pittman (2015). By definition, we have $R_0 = 100\%$, and the aspirational goal fixed at date 0 becomes affordable as soon as $R_t$ hits the threshold $\delta_{\text{asp}}$. In a nutshell, the retirement problem can be summarised as the need to have access to a retirement investment solution that can secure a minimum funded ratio (say 80%) while generating a fair probability of achieving a target funded ratio (say 130%).
3. Optimal Investment Policy with a Target Replacement Income Level
3. Optimal Investment Policy with a Target Replacement Income Level

In this section, we derive optimal strategies that maximise the probability of reaching a replacement income goal in the presence of a minimum replacement income level constraint. These results complement prior work by Browne (1999) and Föllmer and Leukert (1999), which we extend to an environment with a stochastic opportunity set, a liability-driven floor level and intermediate contributions.

3.1 The Model

The minimum level of wealth required at date $T$ to secure the essential goal is $G_T = \delta_{ess} r_{max,0} \beta_T$, and the minimum wealth needed to achieve the aspirational goal at date $T$ is $KGT$, where $K = \delta_{asp}/\delta_{ess}$. More generally, we let $G_t = \delta_{ess} r_{max,0} \beta_t$ denote the price at date $t$ of the annuity that pays an annual income of $\delta_{ess} r_{max,0}$ in retirement.

The problem is to maximise the probability of reaching $KGT$ while staying above the floor $GT$. We solve this problem in a continuous-time framework, where the investor has access to $N$ risky assets whose prices evolve as:

$$\frac{dS_{it}}{S_{it}} = \mu_{it} dt + \sigma_{it}' dz_t,$$

where $z$ is a standard $N$-dimensional Brownian motion that generates all the uncertainty in the economy (so that the filtration $(\mathcal{F}_t)$ is the one associated with the process) and the drifts $\mu_i$ and the volatility vectors $\sigma_i$ are progressively measurable processes. We assume that a locally risk-free asset also exists, which earns the continuously compounded risk-free rate $r$. We finally assume that markets are complete, so that a unique equivalent martingale measure exists (Harrison and Kreps, 1979), and the unique state-price deflator $M$ is given by:

$$M_t = \exp \left[ -\int_0^t \left( r_s + \frac{||\lambda_s||^2}{2} \right) ds - \int_0^t \lambda_s' dz_s \right],$$

where $\lambda_s = \sigma_s (\sigma_s'^{-1} [\mu_s - r_s 1])$, $\sigma_S$ being the volatility matrix of the risky assets, $\mu_S$ their expected return vector and $1$ a conforming vector of ones. $|| \cdot ||$ denotes Euclidian norm.

3.2 Probability-Maximising Strategy

We consider the general case where the investor makes an annual contribution $y$ to the portfolio. Let $\mathcal{F}$ be the set of all year ends between dates $0$ and $T$. The intertemporal budget constraint reads:

$$dW_t = r_t W_t dt + \sum_{i=1}^N W_t w_{it} \left[ \frac{dS_{it}}{S_{it}} - r_t dt \right]$$

$$+ y_{t\in\mathcal{F}},$$

where the $w_{it}$ are the proportions of wealth allocated to the risky assets. We stack them in the vector $w$. We let $Y_t = y \alpha_t$ be the present value of the contribution taking place between dates $0$ and $T$.

The optimisation program reads:

$$\max_{w} \mathbb{P}[W_T \geq KGT],$$

subject to $W_T \geq GT$ and (4).

A similar problem was solved by Browne (1999) in a setting with deterministic risk.
3. Optimal Investment Policy with a Target Replacement Income Level

and return parameters, and by Föllmer and Leukert (1999), who derive the optimal payoff in a setting with stochastic parameters but with no floor and no non-financial income. Proposition 1 gives the optimal payoff and strategy in a more general setting.

The solution to (5) involves the maximum Sharpe ratio (MSR) portfolio, defined as the portfolio that maximises the (squared) instantaneous Sharpe ratio:

$$w_{\text{MSR},t} = \frac{\sigma_t^{-1}\lambda_t}{1'\sigma_t^{-1}\lambda_t}$$

(6)

The Sharpe ratio and the volatility of the MSR portfolio satisfy the relation:

$$\frac{\lambda_{\text{MSR},t}}{\sigma_{\text{MSR},t}} = 1'\sigma_t^{-1}\lambda_t.$$  

This portfolio is closely related to the growth-optimal policy, defined as the one that maximises the expected log return to wealth at horizon $T$:

$$w_{\text{go}} = \arg\max_w E \left[ \ln \frac{W_T}{W_0} \right].$$

Long (1990) shows that $w_{\text{go},t} = \sigma_t^{-1}\lambda_t$, so the growth-optimal strategy is a combination of the MSR portfolio and the risk-free asset, a leveraged combination for reasonable parameter values. For an initial wealth $W_0$, it generates the payoff

$$W_{\text{go},T} = \frac{W_0}{M_T}.$$  

As will appear from Proposition 1, the solution to (5) also involves the "annuity-replicating portfolio", which replicates the annuity price $\beta_t$. The assumption of market completeness ensures that there exists a dynamic combination of the risky assets that replicates a deferred annuity. Again, longevity risk is not easily tradeable, and annuities will be needed to ensure market completeness. Again, we may instead consider a simpler setting where investors require replacement income for a fixed period of time in retirement, in which case the annuities are not needed and the "annuity replicating portfolio" is a deferred bond ladder.

We denote the weights in risky assets of this strategy with $w_{\beta,t}$, and we let $\lambda_{\beta,t}$ and $\sigma_{\beta,t}$ be the Sharpe ratio and the volatility of the annuity. Because it is perfectly correlated with the value of the annuity that finances the essential goal, this portfolio can be called a "goal-hedging portfolio" (GHP). Finally, another building block of the optimal policy will be the accumulation bond, which can be replicated with a dynamic allocation strategy $(w_{\alpha,t})_{0 \leq t \leq T}$ in the risky assets.

To write down the optimal strategy, we introduce an equivalent martingale measure $Q^\beta$ associated with a change of numeraire, as in Rouge and El Karoui (2000). The new numeraire is the annuity, so the Radon-Nikodym density of $Q^\beta$ is:

$$\frac{dQ^\beta}{d\mathbb{P}} = \frac{M_T\beta_T}{\beta_0}.$$  

Denoting the indicator function of an event $E$ with $\mathbb{1}_E$, we have:

Proposition 1 (Probability-Maximising Payoff and Strategy) Assume that:

- Markets are complete;
3. Optimal Investment Policy with a Target Replacement Income Level

• There exists a constant $h$ such that

$Q\beta(E_0) = \frac{1 - \delta_{ess}}{\delta_{asp} - \delta_{ess}},$

where $E_0 = \{MT_T \leq h\beta_0\}.^2$

Then, the optimal terminal wealth in (5) is:

$W_T^* = G_T + \left(\frac{\delta_{asp}}{\delta_{ess}} - 1\right) G_T 1_{E_0}.$

Assume in addition that $\lambda_{MSR,t}, \lambda_{\beta,t}$ and $\sigma_{\beta,t}$ are deterministic functions of time. Then:

• The optimal strategy is:

$w_t^* = \varphi_t \frac{\lambda_{MSR,t}}{\sigma_{MSR,t}} w_{MSR,t} + \left(1 + \frac{Y_t}{W_t^*} - \varphi_t\right) w_{\beta,t} - \frac{Y_t}{W_t^*} \eta_{t,T},$

where:

$\varphi_t = \frac{\delta_{asp} - \delta_{ess}}{\eta_{t,T} R_t} n \left[N^{-1}\left(\frac{R_t - \delta_{ess}}{\delta_{asp} - \delta_{ess}}\right)\right],$

$n$ and $N$ being respectively the probability distribution and the cumulative distribution functions of the standard normal distribution, and $\eta_{t,T}$ being the conditional standard deviation of $\log[W_{\beta,t}/\beta_t]:$

$\eta_{t,T} = \sqrt{\int_t^T \left[\lambda_{MSR,s}^2 + \sigma_{\beta,s}^2 - 2\sigma_{\beta,s,\lambda_{\beta,s}}\right]^2 ds};$

• The maximum success probability conditional on the information available at date $t$ is:

$P_t(W_T^* \geq KG_T) = N\left[N^{-1}\left(\frac{R_t - \delta_{ess}}{\delta_{asp} - \delta_{ess}}\right) + \eta_{t,T}\right].$

Proof. See Appendix A.1.

3.3 Properties of Optimal Strategy

Proposition 1 highlights several important features of the optimal strategy. The first result is that under general assumptions on the opportunity set, the optimal payoff can be statically replicated by a long position in the GHP, which generates a payoff equal to $G_T$, and in a long position in a digital option that pays $KG_T - G_T$ or zero. As a result, the terminal wealth can take only two values, $G_T$ and $KG_T$.

Second, the general expression for the optimal payoff gives general qualitative indications on the composition of the optimal portfolio. Being a function of $M_t$ and $G_T$, the optimal strategy involves dynamic trading in the MSR portfolio and the GHP. As a non-linear function of $M_t$ and $G_T$, it also involves a series of portfolios hedging the risk factors that affect the volatility of the ratio $M/G$. By Ito’s lemma, the volatility vector of $M/G$ is $\sigma_{\beta,t} - \lambda_{\beta}$ and the scalar volatility is the norm of this vector

$\|\sigma_{\beta,t} - \lambda_{\beta}\| = \sqrt{\lambda_{MSR,t}^2 + \sigma_{\beta,t}^2 - 2\sigma_{\beta,t,\lambda_{\beta,t}}}. $

When $\lambda_{MSR,t}, \sigma_{\beta,t}$ and $\lambda_{\beta,t}$ are deterministic functions of time, the hedging portfolios vanish. In this case, the dynamic replication strategy for the optimal payoff is a combination of the MSR portfolio, the GHP, the accumulation bond and cash, with coefficients that can be computed in closed form. The role of the accumulation bond is to cancel the implicit long position in the coupon bond that the investor holds through the scheduled contribution stream.

Note that the GHP and the MSR components can also be found in more...
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standard expected utility maximisation programmes. For instance, the desire to enjoy performance while securing an essential level of income could be captured in the maximisation of the expected utility of $W_T$, or $E[U(W_T - G_T)]$, subject to the constraint $W_T \geq G_T$. Teplá (2001) and Deguest, Martellini, and Milhau (2014) show that the optimal strategy has the form of option-based portfolio insurance, involving both the annuity-replicating and the MSR portfolios. A variant of this program is the maximisation of $E[U(W_T - G_T)]$, where $U$ is a utility function such that marginal utility grows to infinity near zero. The latter property guarantees that $W_T > G_T$ with probability 1 at the optimum. By writing the first-order optimality conditions, it can be verified that the optimal terminal wealth is $G_T + U'^{-1}([\eta M_T])$, where $U'^{-1}$ is the inverse function of marginal utility and $\eta$ is a constant. Thus, the optimal payoff is a non-linear function of $M_T$ and $G_T$, as it is in Proposition 1, so the optimal strategy combines the same building blocks.

The third important property of the optimal strategy is that the allocation to the MSR portfolio depends on the current funded ratio: it approaches zero as the quantity

$$N \approx -\left(\frac{R_T - \delta_{ess}}{\delta_{asp} - \delta_{ess}}\right)$$

goes to minus or plus infinity, that is when $R_T$ approaches $\delta_{ess}$ or $\delta_{asp}$. The property of the allocation to the performance building block shrinking to zero as current wealth approaches a floor is one that is shared with portfolio insurance strategies (see Black and Perold (1992) for constant proportion portfolio insurance and Teplá (2001) for option-based portfolio insurance). This requirement allows investors to avoid breaching the floor. The fact that the risk taking vanishes when wealth approaches a cap corresponds to a totally different concern. Indeed, if $R_T$ could exceed $\delta_{asp}$ at some intermediate date, there would be a positive probability for the final $R_T$ to do so as well, but this would entail a loss in the probability of reaching the objective. Indeed, according to Proposition 1, the optimal choice is to dedicate any dollar invested in excess of $G_0$ to increasing the probability of reaching $\delta_{asp}$ rather than trying to exceed that level. In other words, it is an optimality motive that causes the MSR allocation to shrink to zero near $\delta_{asp}$ so as to secure the aspirational goal as soon as it is reached.

In practice, probability-maximising strategies have shortcomings that are obstacles to their adoption in a retail money management situation, where implementation at a reasonable cost and limited customisation are key criteria. Indeed, aspirational levels may exhibit large variations across individuals, so that an investment policy defined with respect to a given level may not be suited for others. Moreover, even if customisation were possible, such policies would raise the problem of dynamically hedging a non-linear payoff, which requires unobservable inputs such as the Sharpe ratios of the MSR and the annuity-replicating portfolios. It is well known that expected returns are difficult to estimate accurately, so the quality of the replication would be plagued by estimation errors. Besides, unreasonably large levels of leverage are potentially required in the implementation of the strategy for reasonable parameter values. Finally, the digital nature of the
probability-maximising payoffs might be difficult to accept for real investors. Indeed, the aspirational goal is either achieved in full or completely missed, and the investor just receives the essential level in the latter case. It is possible that investors who are anxious about their retirement will perceive such strategies as gambles, and that they will prefer more widespread distribution of replacement income, even if it comes at the expense of a decrease in the probability of success. This concern could be mitigated by penalising deviations from the goal in the objective function, but adding such constraints would add to the complexity of the optimisation problem without alleviating the concern over implementation constraints.
4. Key Requirements for Scalable Retirement Investment Funds
4. Key Requirements for Scalable Retirement Investment Funds

In this section, we turn to the core question of this paper, which is the design of implementable forms of retirement investment funds that can meet the needs of a large cross-section of investors in a scalable manner, that is through a number of funds \( K \) that is much smaller than the number of individuals \( N \). In this section, we analyse the properties that a fund must satisfy to fulfil these conditions.

4.1 The Population of Investors

In the previous section, we focused on a single investor. We now consider a population of \( N \) individual investors \( i = 1, \ldots, N \), who can differ in the following dimensions:

- D1. Gender;
- D2. Retirement age;
- D3. Retirement date;
- D4. Start date of the accumulation phase;
- D5. Initial contribution level (at start date) and annual scheduled contribution;
- D6. Essential (D6a) and aspirational (D6b) goals expressed as fractions of the initially affordable income.

Parameters D2 and D3 determine the current age, so that D1, D2 and D3 collectively determine the term structure of survival probabilities and annuity prices for each investor. If the focus is on replacement income for a fixed number of years in retirement, then the gender specification becomes irrelevant since the corresponding GHP is then a bond ladder and not an annuity. Parameters D5 quantify the saving capacity of the individual and how much he/she is willing to invest in a retirement investment fund. Taken together, parameters D1, D2, D3 and D5 determine the affordable income at the entry date, as a function of the prevailing term structure of interest rates. The addition of D6 allows one to fix the essential and aspirational goal levels.

We adopt the following notation. For investor \( i \) starting to accumulate at date \( s \) and planning to retire at date \( T \), the initial capital is denoted by \( W_{is} \) and the scheduled annual contribution is denoted by \( y_i \) (to alleviate notation, we do not append a subscript \( i \) to the investor-specific parameters \( t \) and \( T \)). We assume that each investor makes an annual contribution at least equal to the scheduled contribution, so that the actual amount of money brought at the end of year \( u \) is \( C_{iu} = y_i + h_{iu} \) where \( h_{iu} \) is nonnegative and represents the unscheduled contribution.

We denote the set of contribution dates posterior to the entry date for individual \( i \) by \( \mathcal{S}_i \); \( y_i \) can be zero in case an individual cannot or does not want to commit to make a fixed cash infusion every year, but the capital invested at the entry date is positive. We let \( \beta_i \) be the annuity price that corresponds to parameters D1 to D3 at a point \( t \) in the accumulation phase, and \( \alpha_i \) be the price of the accumulation bond, which is determined by parameters D2 and D3.

4.2 Securing Investors’ Essential Goals

Individual \( i \) invests in a fund launched at date \( 0 \) with an inception value \( X_0 = \$1 \) and a maturity date greater than or equal to \( T \), so that the fund is available over the entire accumulation period. Since a fixed amount of money \( y_i \) is expected to flow into the fund every year, we assume that the fund value itself jumps by the amount \( C = cX_0 \).
4. Key Requirements for Scalable Retirement Investment Funds

at the end of each year. If \( C = 0 \), the value is continuous. Let \( u \) be the first annual contribution date after the entry date, \( s \). Given that \( X_u = X_{u-} + C \) and \( W_{iu} = W_{iu-} + C_u \) the post-contribution wealth is:

\[
W_{iu} = \frac{W_{is}}{X_s} X_u + \left( y_i + h_{iu} - \frac{W_{is}}{X_s} C \right).
\]

Let us assume that the investor's contributions satisfy:

\[
y_i = \frac{W_{is}}{X_s} C, \tag{7}
\]

which implies that:

\[
\frac{W_{iu}}{X_u} = \frac{W_{is}}{X_s} + \frac{h_{iu}}{X_u}.
\]

By inductive reasoning, we have, for all \( t \geq s \):

\[
\frac{W_{it}}{X_t} = \frac{W_{is}}{X_s} + \sum_{u \in G_i \atop u \leq t} h_{iu}, \tag{8}
\]

where the sum is over all dates \( u \) such that \( u \leq t \). Condition (7) is imposed in order to ensure that if no exceptional contributions are made \( (h_{iu} = 0) \), then the number of shares of the fund is constant.

An investor's essential goal is to preserve a fraction \( \delta_{ess,i} \) of his/her initial purchasing power, a condition that should not only be satisfied at date \( T \) but also at any date \( t \) before \( T \). Indeed, the individual may decide to exit the fund any time before retirement and wants to rely on a replacement income of \( \delta_{ess,i} \gamma_{is} \alpha_{is} - \delta_{ess,i} \gamma_{is} \alpha_{is} \) at least in this event. This constraint is mathematically equivalent to:

\[
\frac{W_{is}}{X_s} X_t \geq \delta_{ess,i} \frac{X_t}{X_s} + \frac{y_i \alpha_{is} \beta_{is}}{\beta_{is}} - \delta_{ess,i} \gamma_{is} \alpha_{is} \frac{X_t}{X_s}, \tag{9}
\]

for all \( t \) in \([s, T]\). Since exceptional contributions cannot be anticipated in advance, this inequality must hold for any choice of \( h_{iu} \geq 0 \), in particular for \( h_{iu} = 0 \). Hence, we must have:

\[
\frac{W_{is}}{X_s} X_t \geq \delta_{ess,i} \frac{X_t}{X_s} + \frac{y_i \alpha_{is} \beta_{it}}{\beta_{is}} - \delta_{ess,i} \gamma_{is} \alpha_{is} \frac{X_t}{X_s},
\]

for all \( t \) in \([s, T]\).

By (7), this is equivalent to:

\[
x_t \geq \delta_{ess,i} \frac{X_s}{X_t} + \frac{y_i \alpha_{is} \beta_{it}}{\beta_{is}} - \delta_{ess,i} \gamma_{is} \alpha_{is}, \tag{10}
\]

for all \( t \) in \([s, T]\). The right-hand side of this equation defines a floor that the fund value must respect at all dates:

\[
F_{it} = \delta_{ess,i} \frac{X_s}{X_t} + \frac{y_i \alpha_{is} \beta_{it}}{\beta_{is}} - \delta_{ess,i} \gamma_{is} \alpha_{is},
\]

for all \( t \) in \([s, T]\). This floor level obviously depends on the gender \( (D1) \), the retirement age \( (D2) \) and date \( (D3) \), the entry date \( (D4) \) and the essential goal \( (D6a) \). On the other hand, it is independent from the aspirational goal and it is to some extent independent from the contribution level since it depends on contributions only through the ratio \( C = X_t Y_t / W_{it} \). Hence, if there does exist a strategy such that (9) holds, the corresponding fund depends on a combination of parameters \( D1, D2, D3, D4, D6a \) and \( C \), and it can accommodate the essential goals of all investors who have these exact parameter values. A strategy that satisfies (9) is designed in Section 5.
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4.3 Relative Max Drawdown Floor

All that was required from the fund in the previous paragraph was to secure the essential goal of an investor entering at date $s$, retiring at date $T$ and satisfying condition (7). To avoid having to launch new funds at every possible date, one may further require that the goals of all investors entering the fund’s life at any date in and satisfying (7) be secured. This means that the inequality in Equation (9) must not only hold for a particular combination of entry date $s$ and retirement date $T$, but for any choice of dates $s$ and $T$ between the fund inception date (date 0) and maturity date.

Let us for a moment restrict the population of investors to whom the scalability constraint applies by assuming that they all have the same gender, retirement age, retirement date ($T$) and essential goal. Thus, they all have the same annuity and accumulation bond prices, so we can remove the subscript $i$ from prices and the essential level of funded ratio $\delta_{ess,i}$, but they can still differ through the entry date, $s$. The scalability constraint (9) becomes:

$$X_t \geq \delta_{ess} \frac{X_s + C\alpha_s}{\beta_s} \beta_t - C\alpha_t,$$

for all $t$ in $[s, T]$ and all $s$ in $[0, T]$, which is equivalent to:

$$X_t \geq \delta_{ess} \left[ \max_{0 \leq \tau \leq s} \frac{X_s + C\alpha_s}{\beta_s} \right] \beta_t - C\alpha_t,$$

for all $s$ in $[0, T]$.             (11)

With $c = C/X_0$, the floor defined by the right-hand side of this equation is:

$$F_t = \delta_{ess} \left[ \max_{0 \leq \tau \leq s} \frac{X_s + C\alpha_s}{\beta_s} \right] \beta_t - cX_0\alpha_t,$$

for all $s$ in $[0, T]$.             (12)

This floor is a “maximum relative drawdown” floor. Indeed, if we let $\tilde{X}_t = (X_t + C\alpha_t)/\beta_t$ be the total fund value (i.e. the fund value plus the present value of future cash additions) expressed in the annuity numeraire, then the floor defined in equation (12) implies that the purchasing power of the total fund value in terms of replacement income in retirement must never lose a fraction greater than $(1 - \delta_{ess})$ of its maximum to date.

The floor (12) still depends on the gender ($D_1$), the retirement age ($D_2$) and date ($D_3$), the essential goal ($D_{6a}$) and the contribution ratio $c$, but it no longer depends on the entry date. Thus, one has to create a fund for each combination of parameters $D_1$, $D_2$, $D_3$, $D_{6a}$ and $C$, and provided we are able to find an investment policy that satisfies (12), the essential goals of all investors who have these parameters will be reached. In Section 5, we show that such a strategy does exist, and we use it to define a fund that is completely described by a combination of $D_1$, $D_2$, $D_3$, $D_{6a}$ and $C$. Hence, for each combination of $D_1$, $D_2$, $D_3$ and $D_{6a}$, there exist multiple funds associated with different values of $c$. We call each of these funds a “$(1, c)$ fund”, where the 1 refers to the initial value (in $)$ and the c refers to the annual cash infusion, equal to $c$. The set of individuals who have the same parameters $D_1$, $D_2$, $D_3$ and $D_{6a}$ as the fund is called the “cohort” of the fund, and we naturally take the fund maturity
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On the investor side, we consider a cohort of individuals who have the same age, the same gender and the same retirement date, but not necessarily the same arrival date and the same contributions. For each individual $i$ starting accumulation at date $t$, $W_i$ is the capital invested at date $t$ and $y_i$ is the scheduled annual contribution, which can possibly be zero if the individual is unable or unwilling to commit to regular cash inflows. We will make a distinction between two cases, depending on whether $W_i$ is greater or less than the quantity

$$D_{it} = \frac{X_{t}^{(1,0)}}{cX_{0}^{(1,0)}}y_i$$

which is the initial wealth of an investor who fits in the $(1, c)$ fund's cohort.

4.4 Elementary Strategies

The ratio $c$ varies almost continuously across investors as a function of their planned contributions, and the range of possible values for $c$ is potentially infinite. For instance, investors in transition (i.e. with short horizons to retirement) tend to have large available wealth compared to their annual contribution capacity. On the other hand, young investors have in general relatively little initial capital to bring. In order to accommodate all profiles, it is ex-ante necessary to create one $(1, c)$ fund for each possible $c$ value, but managing so many funds could be costly and would almost annihilate the benefits of scalability, as there would be hardly more than one client for each fund (the one client corresponding exactly to the very $c$ of the fund).

Fortunately, we show that it is possible to ensure scalability of funds with respect to contribution levels by having only two elementary funds denoted respectively by $(1, 0)$ and $(1, c)$ one for a given value of $c$. Specifically, we consider a $(1,0)$ fund and a $(1, c)$ one with $c \neq 0$, which have the same gender and date and age characteristics. The value of $c$ is fixed, and can be for instance $1$. We denote the fund values with $X^{(1,0)}$ and $X^{(1,c)}$. As we show later, a suitable combination of these two elementary strategies allows the essential goals of a large class of individuals to be reached whatever their profile of contribution streams.
4. Key Requirements for Scalable Retirement Investment Funds

The following proposition gives a detailed expression for the wealth attained with this combination strategy and shows that the essential goal of each individual is attained.

**Proposition 2 (Combination of Funds With Large Initial Capital)** When \( W_i \geq D_{it} \) and \( C_{iu} \geq y_i \) for all \( u \) in \( \mathcal{G}_i \), the wealth achieved by combining the \((1,0)\) and the \((1, c)\) funds is, for \( s \geq t \):

\[
W_{is} = \left[ \frac{W_{it} - D_{it}}{X_t^{(1,0)}} + \sum_{u \in \mathcal{G}_i, u \leq s} \frac{C_{iu} - y_i}{X_t^{(1,0)}} \right] X_s^{(1,0)} + \frac{D_{it}}{X_t^{(1,c)}} X_s^{(1,c)}.
\]

and the final affordable replacement income satisfies:

\[
\max_{t \leq s \leq T} \frac{W_{is}}{\beta_s} \geq \delta_{ess} \max_{t \leq s \leq T} \frac{W_{is} + \frac{W_{it}}{D_{it}} y_i \beta_s}{\beta_s}.
\]

**Proof.** See Appendix A.2.

If the investor makes exactly the required contribution every year, then the numbers of shares held in both elementary funds are constant, and the combination strategy is thus a buy-and-hold allocation to the elementary funds. More generally, the individual purchases new shares of the \((1,0)\) fund during accumulation if his/her contribution exceeds the scheduled contribution, but he/she keeps the same number of shares of the \((1, c)\) block. A practically important point is that he/she never sells any of the funds, an operation that could entail deadweight costs such as exit fees or taxes on dividends or capital gains. The second part of Proposition 2 shows that the individual enjoys the same guarantee level as would be enjoyed by investing in a hypothetical \((W_i, y_i)\) fund that would be launched at date \( t \). This customised fund may not exist but a suitable combination of the available \((1,0)\) and \((1, c)\) funds leads to the same replacement income protection levels.

**4.4.2 “Low” Initial Capital**

When \( W_i < D_{it} \), the former strategy cannot be implemented since it would imply shorting the \((1,0)\) fund and taking a levered position in the \((1, c)\) one. By “continuity” with the previous strategy, the allocation rule at date \( t \) is now to invest everything in the \((1, c)\) fund. Next, at each end of year \( u \), the contribution is broken down as:

\[
C_{iu} = y_i + C_{iu} - y_i = \frac{W_{it}}{D_{it}} y_i + \left[ -\frac{W_{it}}{D_{it}} y_i + C_{iu} - y_i \right].
\]

Only the first term is treated as a regular contribution, and is therefore allocated to the \((1, c)\) fund, while the second term is treated as an excess contribution that is invested in the \((1,0)\) fund.

**Proposition 3 (Combination of Funds With Low Initial Capital)** When \( W_i < D_{it} \) and \( C_{iu} \geq y_i \) for all \( u \) in \( \mathcal{G}_i \), the wealth achieved by combining the \((1,0)\) and the \((1, c)\) funds is, for \( s \geq t \):

\[
W_{is} = \sum_{u \in \mathcal{G}_i, u \leq s} \left( C_{iu} - \frac{W_{it}}{D_{it}} y_i \right) \frac{1}{X_t^{(1,0)}} X_s^{(1,0)} + \frac{W_{it}}{X_t^{(1,c)}} X_s^{(1,c)}.
\]

and the final affordable replacement income satisfies:

\[
\max_{t \leq s \leq T} \frac{W_{is}}{\beta_s} \geq \delta_{ess} \max_{t \leq s \leq T} \frac{W_{is} + \frac{W_{it}}{D_{it}} y_i \beta_s}{\beta_s}.
\]

**Proof.** See Appendix A.3.
4. Key Requirements for Scalable Retirement Investment Funds

At the entry date, the lower bound on replacement income that can be announced to the individual is

$$\delta_{ess} \frac{W_iD_{it}}{\beta_t},$$

a quantity that is less than $\delta_{ess}\alpha_{i,t}$ because we are precisely in the case where $W_i < D_{it}$. Hence, there is no guarantee that the essential goal is reached. This is due to the fact that part of the scheduled contribution is actually treated as exceptional contribution rather than expected contribution, and this part is therefore not included in the computation of the lower bound. To be able to offer the same guarantee to all investors, it is necessary to take sufficiently large $c$. $c = 1$ is a reasonable choice because at date 0, the constraint $W_0 \geq D_0$ means that $W_0 \geq y_p$ and it is natural to expect or require that the periodic contributions do not exceed the initial one.
4. Key Requirements for Scalable Retirement Investment Funds
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The purpose of this section is to design strategies for retirement investment funds that allow the scalability conditions (9) or (11) to be satisfied. A fund that satisfies (9) secures the essential goals of individuals who enter at date $s$ and plan to make a contribution $y_i = W_i X_i / X_0$ every year. For a fund that satisfies (11), the entry date can be any date between inception and maturity. In what follows, we first present the model price for annuities, which plays an important role in the design of the strategies, and we then describe the performance and hedging building blocks that the funds are invested in and the allocation policy to these building blocks.

5.1 Modelling Annuity Prices

The goal is to allow investors to achieve minimum or target levels of replacement income in retirement. In principle, this implies that their wealth at retirement should be sufficient to allow them to buy the corresponding amount of annuities at retirement. In what follows, we explain how to obtain model prices and dynamic hedging portfolios for these annuities, which plays an important role in the design of the strategies, and we then describe the performance and hedging building blocks that the funds are invested in and the allocation policy to these building blocks.

5.1.1 Cash Flows

To price the annuity, we must first specify its cash flows. To protect the actual purchasing power in terms of consumption goods, the annuity payments should be indexed on realised inflation, which requires the use of indexed bonds. A mid-term approach between perfect indexation and no indexation at all is to take into account a cost of living adjustment (COLA) by having the cash flows grow at a predefined rate $\pi$ that represents the annual expected inflation rate over the period $[0, T_{\text{max}}]$. Thus, the cash flows taking place on dates $T + 1, \ldots, T_{\text{max}}$ are fixed in nominal terms and are equal to $(1 + \pi)^{T+1}, \ldots, (1 + \pi)^{T_{\text{max}}}$. This amounts to implicitly choosing date 0 as the reference date for indexation, but as we shall see, the weights and the values of a $(1, c)$ fund do not depend on the level of annuity prices, but on the return on annuities: thus, the choice of a specific reference indexation date is irrelevant.

We adopt the following standard model for the actuarially fair price of an annuity at date $T$:

$$\beta_T = \sum_{s=T+1}^{T_{\text{max}}} B(T, s) H(T, T, a, s)(1 + \pi)^s,$$

where $B(T, s)$ is the discount factor applied at date $T$ to cash flows occurring at date $s$, and $H(t, T, a, s)$ is the probability for an individual aged $a$ at date $T$ to live until date $s$ at least, evaluated at date $t$. In general, insurers do not sell annuities at the actuarially fair price. The ratio of the actual price to the actuarial price represents a “loading factor”, which we treat as constant through time, and which effectively depends on the pricing policies of carrier providers. The assumed value for the loading factor is irrelevant for the rebalancing and the valuation of $(1, c)$ funds since it only impact annuity prices, but has no effect on annuity returns.
5. Designing Retirement Investment Funds

5.1.2 Survival Probabilities
Survival probabilities at date $T$ are in principle not known ex-ante. Uncertain changes in mortality can be represented with a stochastic model, such as that of Lee and Carter (1992), in which the term structure of probabilities by age is driven by a single factor. Unexpected longevity risk cannot be hedged with fixed income instruments, since it mainly depends on factors exogenous to financial markets, such as medical innovation, incidence of accidental causes of death, lifestyle, etc. Hence, we adopt an intermediate approach, as we do for inflation, which involves making a provision only for the expected part of longevity evolution. Thus, future probabilities are treated as deterministic quantities, which are known as of date 0. Appendix B.2 gives details on the computation of survival probabilities from the mortality table and the adjustment factor.

With these assumptions, the no-arbitrage price of the annuity at date $t \leq T$ is:

$$\beta_t = \mathbb{E}_t \left[ \frac{M_T}{M_t} \beta_T \right] = \sum_{s=T+1}^{T_{\text{max}}} B(t, s) H(T, T, a, s)(1 + \pi)^s.$$  

(14)

5.1.3 Differences With Respect to Market Quotes
It must be emphasised here that (14) is a model price for annuity contracts, which may not coincide with the prices quoted by carrier providers for the fund investors. The first reason is that investors will not have the exact same age characteristics as the reference individual. Gender is not an issue here because there are only two possible values and each individual can be assigned to a fund of the same gender. But age and retirement date are continuous variables which vary from one individual to the other. As a result, an investor who does not exactly fit in the fund’s cohort and invests all his/her money in the annuity-replicating portfolio will not have a strictly constant purchasing power in terms of annuities.

Second, even for members of the cohort, the model price may be different from a market price because there is no unique quote for a deferred annuity even when the current age, the deferral period and the gender are specified. Prices and loading factors vary across providers, thereby reflecting the use of different pricing models and/or the application of different market practices. Differences in ratings can also play a role, with low-rated companies having to promise higher payouts in order to compensate for their lower creditworthiness. Numbers illustrating the heterogeneity of pricing policies across insurers are given in Mitchell, Poterba, and Warshawsky (1999). The only way to reconcile the model and the market prices would be to let the loading factor vary over time and over providers in an unrestricted way.

5.2 Building Blocks
As the probability-maximising strategies, the funds will combine three building blocks: a "performance-seeking portfolio" (PSP), taken as a proxy for the MSR portfolio; the goal-hedging portfolio (GHP), which replicates the model price of an annuity; and the accumulation bond.

5.2.1 Performance-Seeking Portfolio
While the theoretical prescription is to build a proxy for the MSR portfolio, the PSP in
5. Designing Retirement Investment Funds

practice is expected to be a well-diversified portfolio aiming to efficiently extract the risk premia from risky assets so as to generate the performance needed to reach aspirational goals. We discuss the construction of this portfolio in more detail in Appendix B.1.

5.2.2 Goal-Hedging Portfolio
This portfolio must replicate the value of a deferred annuity paying one dollar per year from the date of retirement until death. It cannot be invested in true annuities because an annuity is a contract issued by an insurance company on a given individual, and thus cannot be part of a fund offered to multiple investors. Hence, the replicating portfolio will take as a target the price of the annuity for the cohort of the fund. We recall that the cohort is defined by a gender, a retirement age $a_{ret}$, a retirement date $T$ and an essential level of funded ratio $\delta_{ess}$. The first three parameters jointly determine the term structure of survival probabilities, since the age at any date $t$ is given by $a = a_{ret} - (T - t)$.

The only source of risk in this price is variation in interest rates, so the practical implementation of the GHP will be done by duration hedging, where the target (modified) duration is:

$$D_{t} = T - t, \quad T = \max(t, 0).$$

Given the long maturity of the cash flows, this duration is potentially long; it is comprised between $T - t$, which is the time to retirement, and $T_{max} - t$, the maximum remaining longevity. Because the duration of available fixed-income instruments is bounded, it may be necessary in practice to set an upper bound on the time to retirement in order to stay within a range of attainable durations.

5.2.3 Accumulation Bond
The accumulation bond pays an annual coupon of $1 during accumulation. Thus, its price is:

$$\alpha = \sum_{s=0}^{T} B(t, s).$$

The only risk factors are the interest rates serving as discount rates, so the replication of the bond is a standard problem of duration hedging, where the target duration is given by:

$$D_{t} = \frac{1}{\alpha} \sum_{s=0}^{T} (s-t)B(t, s).$$

It is at most equal to the time to retirement, so if the duration of the annuity-replicating portfolio is attainable, the duration of the accumulation bond will also be attainable.

5.3 Strategies for “Customised” Funds
By a customised fund, we mean a fund designed for individuals entering at a specific date, and described by the parameters $D_1, D_2, D_3, D_4, D_6a$ and $c$. For notational simplicity, we assume that the fixed entry date is $s = 0$ and we drop the individual index $i$, so that the floor to protect is, by Equation (10):

$$F_t = \delta_{ess} X_t + C\alpha_s \beta_t - C\alpha_t. \quad (15)$$

We call it a "simple floor" because it does not contain the maximum operator. Let $X_t = X_t + C\alpha_t$ be the total value...
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of the fund. The constraint $X_t \geq F_t$ is equivalent to:

$$X_t \geq \delta_{ess} \frac{X_0}{\beta_0} \beta_t.$$  \hspace{1cm} (16)

Let $F_t$ be the right-hand side. In order to ensure that $X_t \geq F_t$ at all dates, we use a dynamic investment rule such that the amount of money allocated to PSP shrinks to zero as $X_t$ approaches $F_t$. This is also a feature of the optimal strategy, where the MSR allocation shrinks to zero as the funded ratio lands on the essential level. The optimal strategy depends on the risk budget $(X_t - F_t)$ in a non-linear way, but the same property can be obtained by specifying the dollar allocation to the PSP as a linear function of the risk budget $(X_t - F_t)$. Thus, the amount invested in the PSP has the form $m(X_t - F_t)$, where $m$ is a potentially time-varying multiplier. The remainder of total value is allocated to the GHP.

Let $S$ be the PSP value. With the previous strategy, the total value evolves as:

$$dX_t = m \left( X_t - F_t \right) \frac{dS_t}{S_t} + \left[ X_t - m \left( X_t - F_t \right) \right] \frac{d\beta_t}{\beta_t}. \hspace{1cm} (17)$$

Given that $X_t - F_t = X_t - F_t$, we obtain the dynamics of the financial value:

$$dX_t = m \left( X_t - F_t \right) \frac{dS_t}{S_t} + \left[ X_t + C \alpha_t - m \left( X_t - F_t \right) \right] \frac{d\beta_t}{\beta_t} - C \sum_0 \alpha_t.$$

Thus, the dollar amounts invested in the PSP, the GHP and the accumulation bond are:

$$\theta_{S,t} = m \left( X_t - F_t \right),$$

$$\theta_{G,t} = X_t + C \alpha_t - \theta_{S,t},$$

$$\theta_{\alpha,t} = -C \alpha_t, \hspace{1cm} (18)$$

with $F_t$ given by (15). These equations define a fully customised fund, which is specific to parameters $D_1, D_2, D_3, D_4, D_5$ and $D_6a$. For $C > 0$, the fund is a long-short one because of the negative position in the accumulation bond. Note that the investment rule only involves observable parameters, which stands in contrast with the optimal probability-maximising strategy that depends upon risky asset expected return and volatility levels.

Another difference between the proposed simplified investment policy and the probability-maximising strategy is that the former is independent from any aspirational goal. This is an advantage from a scalability perspective, as some investors may not pursue a well-defined target. On the other hand, as explained in Section 3, maximising the probability of reaching a goal by retirement requires securing it as soon as it is reached. Hence, an investor with a goal $\delta_{asp}$ should implement a stop-gain rule and exit the fund at the first date the funded ratio reaches $\delta_{asp}$. At this date $t$, wealth satisfies:

$$W_t \geq \delta_{asp} ri_{max,i,0} \beta_t - y_i \alpha_t.$$  

To make sure that the aspirational goal is secured at date $T$, it suffices to adopt a strategy that yields a constant funded ratio between dates $t$ and $T$. This can be done by taking a buy-and-hold long position in the annuity that pays $ri_{max,i,t}$ in retirement and a short position in the bond that pays
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\( y_i \), during accumulation. Given that the scheduled contributions \( y_i \) continue until date \( T \), the wealth at any date \( s \) after the success time \( t \) is:

\[
W_{is} = ri_{\max,i,t}\beta_s - y_i\alpha_s,
\]

so the affordable replacement income is constant and the funded ratio remains equal to \( R_t \).

5.4 Strategies for \((1, c)\) Funds

In a \((1, c)\) fund, the dependency with respect to the entry point is removed, leaving the parameters be specified as \( D1, D2, D3, D6a \) and \( c \). The floor to protect is the maximum relative drawdown floor in Equation (12), and this condition is equivalent to:

\[
\bar{X}_t \geq \delta_{\text{ess}} \left[ \max_{0 \leq s \leq t} \frac{X_s}{\beta_s} \right] \beta_t,
\]

for all \( t \) in \([0, T]\).

It has the same form as (16), but the initial ratio \( X_0/\beta_0 \) is replaced by the maximum. We adopt the same dynamic investment strategy as for the customised fund, that is (18), but the floor is now given by (12). We note that the rebalancing weights are unchanged upon multiplication of all annuity prices by the same constant. Hence, they are invariant to the choice of the loading factor and the reference date for indexation.

The most important property of \((1, c)\) funds is that they satisfy the essential goals of all members of their cohort. This follows from the discussion in Section 4.3 once it has been verified that the fund value satisfies \( X_t \geq F_t \), or equivalently \( \bar{X}_t \geq \bar{F}_t \). This result itself follows from the work of Cvitanic and Karatzas (1995) and Elie and Touzi (2008), and we reproduce their argument in the Appendix for the sake of completeness.

Proposition 4 (Scalability of \((1, c)\) Funds)

A \((1, c)\) fund secures the essential goal of any member of its cohort.

**Proof.** See Appendix A.4.

An important special case is \( c = 0 \). \((1,0)\) funds are intended for investors who have no scheduled contributions beyond the initial contribution. These individuals can still bring additional dollars if they wish, but the \((1,0)\) fund will only protect a fraction of their affordable income measured in terms of financial wealth. In fact, as will be explained in Section 6, there are practical issues affecting \((1, c)\) funds with \( c \neq 0 \) that do not apply to \((1,0)\) funds. In particular, one clear advantage of taking \( c = 0 \) is that the short position in the accumulation bond vanishes.

With the above strategies, there exists a simple relationship between the values of \((1,0)\) and \((1, c)\) funds for any \( c \). Indeed, let \( \bar{X}^{(1,c)} \) be the value process of a \((1, c)\) fund and \( \bar{X}^{(1,0)} \) be the total value, and let \( Q^{(1,c)}_t = \max_{0 \leq s \leq t} \left( \frac{X^{(1,c)}_s}{\beta_s} \right) \) be the running maximum of the total value in the annuity numeraire. By Equation (17), we have:

\[
dX^{(1,c)}_t = m \left( \frac{X^{(1,c)}_t - \delta_{\text{ess}} Q^{(1,c)}_t \beta_t}{\beta_t} \right) \frac{dS_t}{S_t} + \left[ X^{(1,c)}_t - m \left( X^{(1,c)}_t - \delta_{\text{ess}} Q^{(1,c)}_t \beta_t \right) \right] \frac{d\beta_t}{\beta_t}.
\]

The dynamics are identical to those of \( X^{(1,0)} \), so \( \bar{X}^{(1,c)} \) and \( X^{(1,0)} \) are proportional to each other:

\[
\bar{X}^{(1,c)}_t = \frac{X^{(1,0)}_0}{X^{(1,0)}_0 \beta_0} \bar{X}^{(1,0)}_t. \tag{19}
\]

\[ \]
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This equation implies that the performance of the (1, c) fund in terms of total value equals the performance of the (1,0) fund in terms of financial value.

5.5 Upside Potential of (1, c) Funds and Floor Resets

A (1, c) fund must reach the aspirational goals with a positive probability, regardless of the entry date. To see that this condition is fulfilled, consider an investor arriving at date \( t \) and making no additional contribution beyond the scheduled contributions. The return on wealth equals the fund return (see (8)), so we have, by Equation (31) in Appendix A.4:

\[
\frac{r_{i \max,i,T}}{r_{i \max,i,t}} = \delta_{\text{ess}} \frac{Q_T}{X_t} + \left(1 - \delta_{\text{ess}} \frac{Q_t}{X_t}\right) \left(\frac{Q_t}{Q_T}\right)^{\delta_{\text{ess}} - 1} \left(\frac{S_T \beta_t}{S_t \beta_T}\right)^m \exp \left[\frac{m(1-m)}{2} \int_t^T \left[\sigma_{S,u}^2 + \sigma_{S,u}^2 - 2\sigma_{S,u}\right] dw\right]
\]

(20)

where \( \tilde{X}_t = X_t / \beta_t \) is the maximum affordable income for an investor who enters the fund at date 0 with $1, and \( Q_t = \max_{s \in [0,t]} \tilde{X}_s \). If the PSP sufficiently outperforms the annuity, we have \( r_{i \max,i,T} \geq \delta_{\text{ess}} r_{i \max,i,t} \), and the aspirational goal is reached.

However, the previous formula shows that the exposure of the affordable income to the stock performance depends on the quantity \( 1 - \delta_{\text{ess}} Q_t / \tilde{X}_t \), which measures the risk budget of the fund at the arrival date. If it is low, then the initial PSP weight is also low, and the fund only captures the annuity performance, implying a low probability of reaching goals higher than \( r_{i \max,i} \). As a result, the evolution of affordable income with the (1, c) fund strongly depends of the entry point, and an investor who arrives when the risk budget is low has much lower chances of reaching aspirational goals than one who enters when the risk budget is high.

Since the relative maximum drawdown floor never decreases over time due to the natural growth in the running maximum, it is likely that investors who arrive late after date 0 are disadvantaged with respect to the early ones in terms of access to the upside. In order to re-create a risk budget, it would be necessary to lower the floor, but this has to be done without impacting the essential goal of existing investors. In order to have the possibility of decreasing the floor while respecting the essential goals of investors who arrived earlier, we modify its definition as follows:

\[
F_t = \delta_t Q_t \beta_t
\]

where \( Q_t \) is the maximum of past affordable income levels and \( \delta_t \) is a technical coefficient required to be greater than \( \delta_{\text{ess}} \) at all dates.

At the fund inception, the coefficient is initiated at a target value greater than \( \delta_{\text{ess}} \) so as to allow for a “first line of defence”, and it can be subsequently decreased, an action which we refer to as a floor reset. Resets take place at rebalancing dates whenever the floor-to-wealth ratio

\[
\gamma_t = \frac{F_t}{X_t}
\]

exceeds a threshold \( \gamma_{\text{max}} \). At inception, we have \( \gamma_0 = \delta_{\text{p}} \) and in the event of a reset, the parameter \( \delta_t \) is decreased so as to bring \( \gamma_t \) back to the target \( \gamma_0 \). But after the reset, \( \delta_t \) must remain greater than \( \delta_{\text{ess}} \) in order to continue to protect the fraction \( \delta_{\text{ess}} \) of the purchasing power of already invested
dollars in terms of annuities. Thus, the post-reset value is:

$$\delta_t = \max \left[ \frac{X_t}{\beta_t Q_t}, \delta_{\text{ess}} \right].$$  \hspace{1cm} (21)

An unlimited number of resets is allowed, as long as $\delta_t$ remains greater than $\delta_{\text{ess}}$. But when $\delta_t = \delta_{\text{ess}}$, no other reset is possible.

Resets can be performed in all $(1, c)$ funds, but each value of $c$ implies a specific set of reset dates. As a consequence, the proportionality between the total values of funds for different values of $c$ (Equation (19)) no longer holds in the presence of resets. Typically, resets will occur after a poor relative performance of the PSP with respect to annuities, because this causes a rise in $\gamma$. They improve the ability of the funds to reach the aspirational goal because they increase the upside potential of the strategy for investors who arrive after inception.

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6. Implementation of \((1, c)\) Funds
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In this section, we discuss two questions related to the implementation of the funds, namely the fund creation rules and the issues raised by the non self-financing fund.

6.1 Creation of New Funds

New \((1, c)\) funds can be launched on a regular basis in order to address the needs of younger cohorts. In addition, new funds can be created to replace “sterilised” or “quasi-sterilised” existing funds. Sterilisation refers to the fact that if and when a fund value reaches the floor value, the PSP allocation falls to zero. If this happens, new dollars invested in the fund (whether brought by investors who arrived before or by new clients) have no growth potential at all. The fund is then fully invested in the GHP, and maintains a constant purchasing power which ensures that the essential goal is satisfied but that the aspirational goal cannot be reached.

For a quasi-sterilised fund, the situation is slightly better because the PSP weight is not exactly zero, which leaves room for outperformance with respect to annuities. However, the potential for performance is very limited, and the fund is hardly different from the annuity-replicating portfolio.

When a fund is sterilised, a possible remedy is to reset the floor according to the rule presented before. If \(\delta\) can be decreased to \(\delta_{\text{ess}}\), this allows the PSP weight to be set to \(x_{S,t} = m(1 - \gamma_0)\), as at the fund inception. But if \(\delta\) is already at \(\delta_{\text{ess}}\) or if the post-reset value of \(\gamma\) is still greater than \(\gamma_{\text{max}}\), then possibilities for resets have been exhausted, and a large outperformance of PSP with respect to annuities is required to recover from this state. In order not to offer funds with an overly low PSP allocation, it is necessary to launch a new fund to replace the existing fund. This fund has the same gender, maturity date, reference age and reference date for indexation characteristics as the existing one, but its inception date is posterior. Formally, a new fund is created at each rebalancing date where the post-reset \(\gamma\) exceeds \(\gamma_{\text{max}}\). Because the PSP weight is \(x_{S,t} = m(1 - \gamma)\), it is equivalent to say that a new fund is started when the dollar allocation to PSP is less than \(m(1 - \gamma_{\text{max}})\) of the fund value. When a new fund is created, investors can still leave their existing dollars in the existing fund. From their perspective, even if no growth in the purchasing power is to be expected, they take advantage of the guarantee of the fund, which is to preserve a fraction \(\delta_{\text{ess}}\) of the replacement income that these dollars can finance. We emphasise that the criterion \(\gamma > 95\%\) to trigger the creation of a fund is a purely mechanical rule that we apply to simulate the new fund creations in the Monte-Carlo exercises that we perform. In practice, the decision to start a new fund will likely depend on other criteria, including the effective demand for the product.

6.2 Limits to the Usage of \((1, c)\) Funds for Non-Zero \(c\)

In theory, \((1, c)\) funds allow investors who commit to regular contributions to protect their total purchasing power, but the use of such funds raises practical issues which we examine here.

6.2.1 Non-Respect of Contribution Schedule

The first difficulty arises when a scheduled contribution is not respected, that is when
6. Implementation of (1, c) Funds

the actual cash infusion is less than the scheduled amount. To see the consequences of this situation, let us consider an individual entering the (1, c) fund at date 0, with a wealth $W_0$, and a scheduled contribution $y_t = W_0 C/X_0$ and assume that at the first end of year $u$, the actual contribution $C_u$ is such that $C_u < y$. At date 0, the individual purchases $n_0 = W_0/X_0$ shares of the fund, and at date $u$, his/her wealth is:

$$W_{iu} = n_0 X_u + C_u - \frac{W_0}{X_0} C < n_0 X_u.$$

Hence, the individual cannot afford the $n_0$ shares of the fund. Intuitively, this is because there is an upward jump in the fund value that is not compensated by a commensurate contribution. As a consequence, the investor must sell shares of the fund, an operation which can trigger fees or taxes. Moreover, at the fault of the investor, the guarantee to secure a replacement income of $\delta_{\text{essrimax},i,0}$ is lost. The magnitude of the sale depends on the difference between $W_{iu}$ and $n_0 X_u$, so it increasing in $C$.

It should be noted that this caveat also applies to new (1, c) funds launched to replace quasi-sterilised funds. If an individual who already owns shares of a (1, c) fund decides to invest new dollars in a new fund in order to enjoy greater upside potential, these dollars are lost for the previous fund, which forces him/her to liquidate shares of this fund. For these reasons, (1, c) funds with positive $c$ are difficult to use unless the investor is strongly committed to respecting the schedule. In case his/her capacity to fulfil the commitment is doubtful, (1,0) funds may be preferred investment supports.

6.2.2 Implementation of Long-Short Positions

Another difficulty is faced by the fund manager, and it is due to the short position in the accumulation bond. In practice, it may not be feasible to short a bond, so another option is to replicate the long-short position in the annuity-replicating portfolio and the bond as a whole rather than to implement both positions separately. Consider the budget equation for the fund:

$$\frac{dX_t}{X_t} = \frac{dS_t}{S_t} + \frac{C_0}{X_t} D_{\alpha,t} \left( 1 + \frac{C_\alpha}{X_t} - x_{S,t} \right) \frac{d\beta_t}{\beta_t} - \frac{C_\alpha}{X_t} \frac{d\alpha_t}{\alpha_t}$$

where $x_{S,t} = \theta_{S,t}/X_t$ is the PSP weight. Assuming that the annuity and bond prices are driven by one risk factor only, we have:

$$\frac{d\beta_t}{\beta_t} = -D_{\beta,t} \, dr_t, \quad \frac{d\alpha_t}{\alpha_t} = -D_{\alpha,t} \, dr_t,$$

where $r$ designates the level of interest rates and the $D$-s are the modified durations. Then, the fund dynamics are:

$$\frac{dX_t}{X_t} = \frac{dS_t}{S_t} + \frac{C_0}{X_t} \left[ \frac{D_{\alpha,t}}{D_{\beta,t}} \left( 1 + \frac{C_\alpha}{X_t} - x_{S,t} \right) D_{\beta,t} \right] dr_t$$

$$= x_{S,t} \frac{dS_t}{S_t} + (1 - x_{S,t}) \frac{dH_t}{H_t},$$

where $H_t = (1 - x_{S,t})X_t$ is the price of a long-short bond, the duration of which is:

$$D_{H,t} = \frac{[(1 - x_{S,t})X_t + C_\alpha X_t D_{\beta,t} - C_\alpha D_{\alpha,t}]}{(1 - x_{S,t})X_t}$$

$$= D_{\beta,t} + \frac{C_\alpha}{(1 - x_{S,t})X_t} [D_{\beta,t} - D_{\alpha,t}].$$

(22)
6. Implementation of (1, c) Funds

The annuity has strictly longer duration than the accumulation bond, because $D_{\beta,t} > T - t$ and $D_{\alpha,t} < T - t$, where $T - t$ is the time to retirement. Hence, the duration of the long-short bond is always longer than that of the annuity. But the main problem is that it is potentially unbounded: if the risk budget is large enough, $x_{S,t}$ approaches 1 while $[D_{\beta,t} - D_{\alpha,t}]$ remains finite, so $D_{H,t}$ grows to infinity. The problem is exacerbated if $c$ is large. For this reason, (1, c) funds with $c \neq 0$ raise implementation issues, so in the numerical analysis in the next section, we will test strategies that use both (1,0) and (1, c) funds and strategies that use only (1,0) funds.
7. Numerical Analysis
7. Numerical Analysis

In this section, we perform a series of Monte-Carlo simulations in order to measure the probabilities of reaching aspirational goals for various investor’s profiles and investment strategies.

The objectives are:
• To specify values for the many parameters involved in the design of (1, c) funds;
• To measure the loss in probability (opportunity cost) associated with the use of suboptimal strategies and with the limited customisation of (1, c) funds;
• To compare (1, c) funds to traditional retirement savings products, such as balanced funds and target date funds;
• To show the positive effects of resets on the time to launch new funds, which is a critical point for fund managers who want to have a reasonably low number of funds;
• To show that new fund creations to replace quasi-sterilised funds are useful, offering new investors high probabilities of success.

7.1 Stochastic Model and Parameter Values

To evaluate probabilities by Monte-Carlo simulations, we need to specify a data-generating process for the returns on the building blocks of the strategies (PSP, GHP and accumulation bond). Parameters related to the risk factors that impact these returns are called “objective”. Also included in this class are the technical parameters used in the rebalancing of the funds, namely \( m, \delta_0 \) and \( \gamma_{\text{max}} \) and the annual indexation rate \( \pi \). On the other hand, “subjective” parameters are specific to each investor.

7.1.1 The Stochastic Model

We deliberately use a very simple stochastic model in order to minimise the number of objective parameters to specify. The PSP value is modelled as a process with constant volatility and Sharpe ratio:

\[
\frac{dS_t}{S_t} = \left[ \sigma S \lambda S \right] dt + \sigma S d\zeta_{St},
\]

where \( \zeta_5 \) is a standard Brownian motion and \( r_t \) is the nominal short-term interest rate. For practical purposes and for the calibration of the parameters \( \sigma_S \) and \( \lambda_S \), \( S \) will be thought of as a stock index.

The short-term rate follows the Vasicek (1977) model:

\[
dr_t = a(b - r_t) dt + \sigma_r d\zeta_{rt},
\]

where \( \zeta_r \) is a standard Brownian motion correlated with \( \zeta_S \), the correlation coefficient being denoted with \( \rho_{Sr} \). The model involves a fourth parameter, which is the price of interest rate risk, \( \lambda_r \). Zero-coupon prices are given by the textbook formula:

\[
B(t, s) = e^{-D(s-t)r_t+E(s-t)},
\]

\[
D(u) = \frac{1 - e^{-au}}{a},
\]

\[
E(u) = \left( h - \frac{\sigma_r \lambda_r}{a} \right) [D(u) - u] + \frac{\sigma_r^2}{2a^2} \left[ u - 2D(u) + \frac{1 - e^{-2au}}{2a} \right].
\]

With Equation (14), it is seen that annuity prices depend on one risk factor, which is \( r_t \).

Of course, this model misses a number of well-established stylised facts on stock
returns, such as volatility clustering (Bollerslev, 1986), predictability in excess returns (Fama and French, 1988; Welch and Goyal, 2008) or large and sudden falls in prices, which cannot be modelled with Brownian increments and require the introduction of jump processes (Jorion, 1988). Similarly, on the term structure side, it is well known that while a single level factor explains most of the common variation in interest rates, some movements of the yield curve can only be accounted for with additional slope and curvature factors. In addition, bond returns exhibit some predictability (Fama and French, 1989; Cochrane and Piazzesi, 2005), and the correlation with stock returns is time-varying (Baele, Bekaert, and Inghelbrecht, 2010). These features could be reproduced by the model by introducing new state variables and they would certainly lead to more realistic scenarios. On the other hand, with a suitable choice of parameter values, the proposed model captures three properties of stock and bond returns that are essential in this paper:

- Over the long run, stocks outperform bonds and annuities, so selecting a positive multiplier in the $(1, c)$ fund strategy (18) should result in higher probabilities of reaching aspirational goals than taking $m = 0$;
- Stocks are more volatile than and are also imperfectly correlated with bonds and annuities, so they induce short-term gap risk, implying that the multiplier $m$ should not be too high;
- Interest rates tend to revert to their long-term means. For instance, starting from the historically low levels of 2016, they are more likely to grow than stay flat or further decrease in the future.

### 7.1.2 Objective Parameters

The Sharpe ratio and the volatility of the PSP are estimated as the long-term values for a broad equity index. We borrow the estimates of Dimson, Marsh, and Staunton (2008), who study the US stock market between 1900 and 2000: the volatility of returns is 19.99% per year, and the Sharpe ratio is obtained by dividing the average excess return over the risk-free rate, namely 7.70%, by the volatility of excess returns, 19.60%. Baele, Bekaert, and Inghelbrecht (2010) find that the time-varying stock-bond correlation estimated over a rolling-window or with a dynamic model (GARCH-MIDAS) takes on positive and negative values, so we set $\rho_{Sr}$ to zero. The initial stock value, $S_0$, is irrelevant as it has no impact on simulated returns; we set it to $1$.

There are five interest rate parameters to specify: $a$, $b$, $\sigma_r$, $\lambda_r$ and the initial value $r_0$. We face a consistency constraint between the zero-coupon prices implied by the Vasicek model and those inferred from the observed yield curve. Indeed, the first date in the simulated paths (date 0) coincides with the date at which the simulation is performed, so it is logical to use market data on interest rates to value zero-coupons at date 0. But at the next simulation date, prices are simulated with the Vasicek model. To ensure a smooth transition between zero-coupon and annuity prices from date 0 to the next date, model-implied prices at date 0 should be as close as possible to observed bond prices. These requirements seem to call for a calibration of interest rate parameters to the current yield curve, but such a procedure is unable to disentangle the historical long-term mean $b$ from the price...
7. Numerical Analysis

of risk \( \lambda_r \); by (23), it can only identify the risk-neutral mean \( b - \sigma \lambda_r / a \). Moreover, by treating all parameters as implicit functions of the current yield curve, it introduces unacceptable instability in the estimates.

To achieve a compromise between consistency with the observed yield curve and parameter stability, we mix historical estimation and calibration to market data as follows. We take the Secondary Market rate on the US 3-month Treasury bill as an observable proxy for the short-term rate, and we estimate \( b \) as the average of end-of-month values over the past 20 years. We fix \( a \) so as to equate the steady-state one-year autocorrelation of \( r \) with the historical one-year autocorrelation of the T-bill rate. Finally, \( \sigma_r \), is chosen so as to match the steady-state volatility of \( r \) and the historical volatility. The model-implied values for the autocorrelation at horizon \( h \) and the volatility in the stationary state are:

\[
\lim_{t \to \infty} \text{Corr}[r_{t-h}, r_t] = e^{-ah},
\]

\[
\lim_{t \to \infty} \mathbb{V}[r_t] = \frac{\sigma_r^2}{2a}.
\]

Figure 1: Observed and model-implied zero-coupon yield curves.

(a) January 4, 2016. (b) January 2, 1996.

The observed curves correspond to the zero-coupon rates computed by the method of Gürkaynak, Sack, and Wright (2007) and are downloaded from the Federal Reserve website. Fitted rates are implied by the Vasicek model.

Another category of objective parameters consists of the technical management parameters, which are used to compute the fund weights at rebalancing dates. Given an essential level \( \delta_{\text{ess}} \), the parameter \( \delta_t \) is completely determined by Equation (21), the initial condition \( \delta_0 \) and the threshold \( \gamma_{\text{max}} \). We set \( \gamma_{\text{max}} = 95\% \) (i.e. the fund is
regarded as quasi-sterilised whenever the floor represents at least 95% of the fund value). We take $\delta_0 = 85\%$, so that $\delta$ starts a bit above the minimum admissible value and there is room for resets. We will show later (see Section 7.6) that by taking $\delta_0$ greater than $\delta_{ess}$ rather than equal to it, one reduces the time to launch a new fund. The choice of the multiplier will be discussed in detail in Section 7.3 and will be done by examining the probabilities of achieving goals.

Finally, the annual indexation rate $\pi$ is fixed following the Consumer Sentiment Survey conducted by the University of Michigan: we use the median one-year expected inflation rate from the month that immediately precedes the fund inception. This corresponds to 2.70% for a fund launched in January 1996 (value in December 1995) and 2.60% for a fund launched in January 2016 (value in December 2015).

### 7.1.3 Subjective Parameters

Subjective parameters relate to the investor and consist of inputs D1 to D6. They define a customised $(1, c)$ fund, where $c$ is the ratio of the annual scheduled contribution to the initial capital, $c = y/W_0$ and the inception date is the beginning of the accumulation phase. As far as the gender is concerned, we only report results for male investors. The only difference between male and female investors is that the latter have longer life expectancy at a given age, which raises annuity prices and lowers affordable income. But the impact on the funded ratio, which is the ratio between the current and the initial values of the affordable income, should be moderate. This leaves us with parameters D2 to D6 to specify.

In the base case, we set the retirement age $(D2)$ to 65 years, which is a reasonable choice in most countries. Some of the individuals may retire earlier or later, but there are no reasons to believe that results would dramatically change with a retirement age of 60 or 70. On the other hand, the length of the accumulation period is expected to have a strong impact through the performance of equities and the mean reversion in interest rates. We will consider initial ages of 35, 45 and 55, which imply accumulation periods of 30, 20 and 10 years respectively. This covers a wide range of investors, from young individuals to individuals in transition towards retirement.

For D4, we consider two inception dates: 4 January 2016 and 2 January 1996. They correspond respectively to low and high interest rates, and given the long duration of annuities, the initial level of interest rates has a large impact on the simulated return to annuities. The relative performance of stocks and annuities is a key driver of the evolution of the funded ratio, and we will assess the effect on success probabilities. D2 and D4 jointly determine the retirement date $(D3)$.

For the ratio $c$ $(D5)$, it is reasonable to set a cap at 1, as the initial contribution includes the scheduled contribution plus possibly some additional contribution. We let $c$ take on five values: 0, 1/50, 1/10, 1/5 and 1. In the first case, the individual has no scheduled contribution. In the second case, the capital invested is 50 times the scheduled contribution. In terms of dollar amounts, this means that if the investor plans to save $10,000 every year in the retirement plan, the initial capital is $5,000,000. The other cases correspond to investors who have less...
money to bring at inception compared to their regular saving capacity. In fact, current age and $c$ are likely negatively correlated across investors, since investors in transition have accumulated more money than the young ones.

The essential goal (D6a) is to keep the affordable income above the floor $\delta_{\text{ess}} ri_{\text{max},0}$ at all times. We set $\delta_{\text{ess}} = 80\%$ for all funds, that is we impose that investors will sacrifice at most 20% of the purchasing power in terms of annuities of any money that they invest in the fund in exchange for upside potential. For the aspirational goal (D6b), we will consider multiple levels, of 110%, 120%, 130%, 140% and 150% respectively, so as to capture different individual preferences.

Once the gender and the current age are fixed, survival probabilities at any horizon can be computed from a mortality table (see Section 5.2.2 for details). It is important to use tables that relate to the annuitant population because annuity buyers tend to be wealthier and healthier than the general population, and as a result live longer on average (Mitchell, Poterba, and Warshawsky, 1999). Such tables are published by the Society of Actuaries, and they are period tables. As Mitchell, Poterba, and Warshawsky (1999), we use basic tables, that is tables that do not include a conservative adjustment of mortality probabilities. In contrast, a “loaded” table is obtained by multiplying the basic probabilities by a factor of less than one, so as to raise annuity prices and take a more prudent stand on mortality. The two most recent basic annuitant mortality tables published by the Society of Actuaries are the Annuity 2000 Basic Table and the 2012 Individual Annuity Mortality Basic Table. They were released in January 1995 and September 2011 respectively. When computing probabilities at date $t$, we use the most recently known table at date $t$: to allow for a lag between the publication of a table and its adoption, we take the 2012 table when $t$ is posterior to 1 January 2012 and the 2000 one otherwise. In each case, we apply a projection scale factor to account for the expected decrease in mortality probabilities: these are Projection Scale G for the 2000 table and Projection Scale G2 for the 2012 table. The documentation of the tables on the Society of Actuaries website indicates that these factors were developed “in conjunction” with the corresponding tables, so it is consistent to use them jointly.

### 7.2 Simulation Protocol

In what follows, we perform Monte-Carlo simulations of the performance of $(1, c)$ funds as well as other strategies. In the first step, we generate 10,000 scenarios for the two risk factors (stock returns and interest rates) from the stochastic model. In the first step, we generate 10,000 scenarios for the two risk factors (stock returns and interest rates) from the stochastic model. Second, we simulate strategies with a quarterly rebalancing. We choose the quarterly frequency because it is commonly adopted by fund managers. For the customised fund and the $(1, c)$ funds, we compute the rebalancing weights with Equation (18), and we impose a no leverage constraint in the PSP, by capping its weight to 100% and imposing a minimum weight of 0%. For each strategy, the probability of reaching the essential goal is the probability of the funded ratio remaining above the essential level of 80% at all times, and the probability of reaching an aspirational goal is the probability of the ratio to hitting the level $\delta_{\text{asp}}$ at least once before retirement. Once the threshold is
7. Numerical Analysis

attained, the investor can secure his/her funded ratio by making a stop-gain decision.

7.3 Choice of Multiplier
Before we present results for (1, c) funds, we must select a value for \( m \). As explained before, the trade-off is between a low \( m \), which limits the access to the performance of the stock market, and a high \( m \), which may imply a substantial probability of gap risk. We thus choose a value by evaluating the probabilities of respecting the essential goal and of achieving various aspirational goals.

7.3.1 Results for (1,0) Funds
Table 1 shows the probabilities for investors initially aged 35, 45 or 55. The essential level is virtually respected for all values of \( m \) less than or equal to 3, but the effects of gap risk are visible for \( m = 4 \) or 5, with a non-zero probability of breaching the floor. In fact, due to the use of the relative maximum drawdown floor, breaching the floor does not always imply a violation of the goal. Indeed, we have \( F_t \geq \delta_{asp} r_{\text{max},0} \beta_t \), so it is possible to have \( X_t < F_t \) without having \( X_t / \beta_t < r_{\text{max},0} \). In other words, the growth mechanism embedded in the floor acts like a buffer that absorbs some of the negative shocks on the fund and saves the funded ratio from falling below the essential level.

For \( m = 5 \), deviations from the goal are rare with the 2016 interest rate parameters but they are no longer negligible if the model is calibrated to the 1996 yield curve, occurring with probabilities lying at 5% and 10%. Indeed, simulated interest rates are higher in the latter case because of the difference in the initial level of rates (see Figure 1), so the simulated performance of annuities is also higher. The same is true for the stock because the model assumes

<table>
<thead>
<tr>
<th>( \delta_{asp} ) [%]</th>
<th>35 Years</th>
<th>45 Years</th>
<th>55 Years</th>
</tr>
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<td>110</td>
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<td>63.1</td>
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</tr>
<tr>
<td>150</td>
<td>52.8</td>
<td>60.5</td>
<td>80.5</td>
</tr>
</tbody>
</table>

Table 1: Effect of multiplier on success probabilities for (1,0) fund.
a constant expected excess return, so the expected return itself is increasing in the level of rates, but the effect is more important for annuities because of their long duration. Hence, it is more difficult for stocks to outperform annuities conditional on the 1996 parameters, and this results in increased gap risk. The problem is more acute for the 35-year and the 45-year investors, who have the longest annuity durations.

At the other end, $m = 1$ is a prudent choice that keeps the fund clear from gap risk, but probabilities of reaching funded ratios above 100% are disappointing, especially for the 45-year old and the 55-year old investors. The 35-year old is less sensitive to this problem because s/he takes advantage of a longer accumulation period, but s/he can still benefit from a higher $m$: in 2016, the probability of reaching 130% is 60.2% with $m = 1$, but it grows to more than 80% with $m = 2$ or 3. Selecting an $m$ greater than 1 is all the more crucial for the 45-year old and 55-year old investors. With $m = 1$, the probabilities of reaching 130% are less than 50%, and they fall below 15% if the more ambitious level of 150% is targeted.

Between the extreme choices $m = 1$ and 5, the values 2, 3 and 4 yield probabilities of less than 2% of missing the essential goal and 70% or more of reaching aspirational levels of 120% and 130%, except for the 55-year old investor. In this case, the probability of attaining 130% lies between 39% and 64%. The exact value of $m$ that maximises the probability depends on the
initial age and the aspirational goal, but it appears to be rather robust to the choice of interest rate parameters. For the targets 120% and 130%, it is 2 for the 35-year old, 3 for the 45-year old and 4 for the 55-year old. The shift towards larger values as the investor ages is explained by a straightforward effect: as the accumulation period shortens, a higher $m$ is needed in order to gain more access to the PSP performance.

While it is feasible to choose one $m$ for each initial age, we adopt the simpler approach of taking the same value for all funds. Since the size of the retirement savings market increases with current age, more weight should be given to the results for 45-year old and 55-year old investors. $m = 3$ gives the best probabilities of reaching 130% or more for the 45-year old one. $m = 4$ is in general optimal for the 55-year old individual, but at the cost of a small probability of missing the essential goal. Hence, we retain $m = 3$ for all (1,0) funds in what follows.

The observation that the optimal $m$ decreases as the accumulation period lengthens suggests that another meaningful option would be to have a time-varying $m$ within each fund, with a lower value for young investors and a more aggressive strategy near maturity. In implementation, it would be perfectly conceivable to have a strategy with a time-varying $m$, starting with say $m = 2$ for ages between 35 and 45, $m = 3$ for ages between 45
7. Numerical Analysis

![Highly customised fund, m = 3.](image)

The investor is a male aged 45 in January 2016 and retiring in 2036 at the age of 65. No contributions take place after the initial one. In Panel (a), the investor follows the optimal strategy that corresponds to the aspirational goal $\delta_{\text{asp}} = 130\%$. With this strategy, the terminal funded ratio has a binary option distribution. In Panels (b), (c), (d), the contribution is invested in a (1,0) fund with a multiplier equal to 1, 3 or 5. In Panel (e), the investment support is the highly customised fund with a simple floor. In Panels (b) to (e), the investor exits the fund at the first date the funded ratio reaches the aspirational level of 130\%.

and 55, and $m = 4$ for ages between 55 and 65. Interestingly, this prescription is at odds with the prescriptions of target date funds with a deterministic glide path, where the stock allocation decreases when approaching maturity.

To visualise the impact of $m$, Figure 2 shows the distribution of the terminal funded ratio for an investor of the (1,0) type who uses a (1,0) fund in which the multiplier is 1, 3 or 5 and who shifts to the GHP as soon as the funded ratio reaches a target at 130\%. Also shown is the distribution achieved with the optimal strategy that corresponds to this aspirational level. It follows from Proposition 1 that the optimal distribution is bimodal with values of 80% and 130%. With the (1,0) fund, the distribution also exhibits a mode around 130%, which is due to the fact that the investor transfers his/her assets from the fund to the GHP as soon as the goal is reached. The terminal funded ratio can exceed 130% because the funded ratio is monitored in discrete time and the decision to switch to the GHP can occur after it has reached 130%. A higher $m$ leads to increased performance, hence to higher funded ratios, but this effect is not unlimited because of the leverage constraint in the PSP, which caps the weight at 100%. With a higher $m$, it is also seen that the distribution becomes more bimodal, and tends to concentrate near 80% and above 130%.

### 7.3.2 Results for (1, c) Funds with $c \neq 0$

Consider now a (1, c) fund with a positive $c$. If its total value is proportional to that of the (1,0) fund (Equation (19)), then any member of the cohort of this fund has the same funded ratio as a member of the (1,0) fund cohort. Hence, success probabilities are identical. But (19) does not hold here for two reasons. First, floors are not reset on the same dates for (1, c) and (1,0) funds. Second, funds are rebalanced in discrete time, and (19) is derived by noting that the dynamics of total value are the same for both funds, but this premise is only true when they are rebalanced continuously (and the floors are not reset). Hence, Table 1 is not identical for a (1,0) and a (1, c) fund.

To examine how it is impacted, we look at the case of the (1,1) fund in Table 2.
7. Numerical Analysis

Table 2: Effect of multiplier on success probabilities for (1,1) fund.

Panel A: Fund launched in January 2016 (in %).

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<td>Essential</td>
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Panel B: Fund launched in January 1996 (in %).

<table>
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</tbody>
</table>

The investor is a male aged 35, 45 or 55 years who starts to accumulate money in January 1996 or 2016. In each January month after the beginning, he saves a dollar amount equal to the capital initially invested. The investment support is a (1,1) fund launched on the initial date with a multiplier of 1, 2, 3, 4 or 5. The essential goal is to preserve 80% of the initial affordable replacement income at all dates, and the aspirational goal is to reach a multiple greater than 100% of this income at the terminal date. A stop-gain decision is made on the first date the funded ratio hits the aspirational level. The table displays the probabilities of reaching the goals.
Investors contribute as much every year as they do initially. There are several similarities with the (1,0) case: failure to reach the essential goal is a rare event (and in fact less frequent than in the (1,0) case even for \( m = 4 \) or 5), the value of \( m \) that maximises the chances of reaching a given target tends to increase with age, and \( m = 1 \) is never the best choice. Having said this, the probability-maximising values of \( m \) tend to be higher than in Table 1: with 2016 parameters, \( m = 4 \) is in general better than \( m = 3 \) for the 45-year old investor, even though it is still the best choice with the parameters corresponding to the yield curve observed in January 1996. Overall, the difference between \( m = 3 \) and \( m = 4 \) is modest, less than 3\%, and for consistency we still take \( m = 3 \) for (1,1) funds.

### 7.4 Opportunity Cost of Suboptimality and Limited Customisation

We now compare various strategies, using varying degrees of customisation and observing their probabilities of reaching aspirational goal levels.

#### 7.4.1 Description of Strategies

There are several alternatives to the probability-maximising strategy, which differ through the degree of customisation. The most customised strategy, S1, uses a fund with the simple floor given by (10). It protects 80\% of the initial purchasing power of the investor who enters the fund at inception date, but no attempt is made to offer the same guarantee to investors who arrive later, who would need to be offered a dedicated fund. Since the floor does not rise as the ratio \( X/\beta \) grows, the risk budget should be larger than with the maximum relative drawdown floor (12) and the upside potential is expected to be higher. The next strategy, S2, uses a (1, c) fund with the investor-specific ratio \( c = y/W_0 \). It is less customised than S1 because the fund is designed to secure the essential goals of investors who arrive at any point in the fund’s life. On the other hand, a dedicated (1, c) fund cannot be created for each individual-specific ratio \( c \), so another opportunity cost arises if this fund is unavailable and investors only have access to (1,0) and (1,1) funds. Then, each individual must combine the two blocks following the rules presented in Section 4. This is strategy S3. Finally, in view of the practical problems posed by (1, c) funds for a non-zero \( c \) (see the discussion in Section 6.2), the fund provider may eventually decide to offer only (1,0) funds. Strategy S4 uses only the (1,0) fund, even for investors with planned contributions. All these strategies have by definition lower probabilities of reaching a given aspirational goal than the optimal policy which serves as a non-investable benchmark, so we are interested in measuring the loss of probability with respect to this theoretical maximum, and the effects of the limited customisation on success probabilities.

For an investor with initial wealth \( W_0 \) and a scheduled annual saving \( y \), the wealth with each suboptimal strategy can be computed as follows:

- With a non self-financing fund (highly customised one or (1, c) fund):

\[
W_t = \frac{W_0}{X_0} X_t,
\]

where \( X \) is the fund value;
7. Numerical Analysis

- With a self-financing fund (the (1,0) one):
  \[ W_t = \left( \frac{W_0}{X_0} + \sum_{u \in \mathcal{F}, u \leq t} \frac{y_u}{X_{u-t}} \right) X_t; \]  \[ (24) \]

- With the combination of (1,0) and (1,1) funds:
  \[ W_t = \frac{W_0 - y X_t^{(1,0)}}{X_0^{(1,0)}} + \frac{y}{X_0^{(1,1)}} X_t^{(1,1)}. \]

For the optimal strategy, we only simulate the terminal wealth, given in Proposition 1:
\[ W_T = \left[ \delta_{\text{ess}} + (\delta_{\text{asp}} - \delta_{\text{ess}}) \mathbb{1}_{E_0} \right] r_{i_{\text{max}},0}\beta_T, \]
where \( E_0 = \{ M_T \beta_T \leq h \beta_0 \} \) and \( h \) is such that \( q^\beta(E_0) = \frac{1 - \delta_{\text{ess}}}{\delta_{\text{asp}} - \delta_{\text{ess}}}. \)

In each case, the investor’s funded ratio is \( R_t = W_t/(ri_{\text{max}},0\beta_t) \). It is remarkable that for the optimal strategy, the terminal ratio does not depend on \( c \), since we have
\[ R_T = \delta_{\text{ess}} + (\delta_{\text{asp}} - \delta_{\text{ess}}) \mathbb{1}_{E_0}, \]
and the definition of \( E_0 \) only involves the random variables \( M_t \) and \( \beta_t \), the initial price \( \beta_0 \) and the goals \( \delta_{\text{ess}} \) and \( \delta_{\text{asp}} \).

7.4.2 Probabilities of Reaching Essential Goals
Table 3 displays the success probabilities.\(^9\) To assess the probabilities of reaching different aspirational goals, we have assumed in this table that individuals do not implement the stop-gain mechanism. By definition, this does not affect the probabilities of reaching the aspirational goals. This can slightly decrease the probabilities for the essential goal, as it is possible in theory for the funded ratio to fall below 80%.

### Table 3: Comparison of optimal strategy and strategies invested in (1, c) funds.

<table>
<thead>
<tr>
<th>Panel A: Probabilities evaluated in January 2016 (in %).</th>
<th>( c = 0 )</th>
<th>( c = 1/50 )</th>
<th>( c = 1/10 )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>S1</td>
<td>S2-S4</td>
<td>S1</td>
</tr>
<tr>
<td>Essential ( \delta_{\text{asp}} ) (%)</td>
<td>100.0</td>
<td>99.9</td>
<td>100.0</td>
</tr>
<tr>
<td>110</td>
<td>99.2</td>
<td>91.8</td>
<td>91.8</td>
</tr>
<tr>
<td>120</td>
<td>97.5</td>
<td>85.0</td>
<td>82.8</td>
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<tr>
<td>130</td>
<td>95.5</td>
<td>79.7</td>
<td>74.1</td>
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<td>93.6</td>
<td>75.5</td>
<td>65.2</td>
</tr>
<tr>
<td>150</td>
<td>91.7</td>
<td>72.4</td>
<td>57.7</td>
</tr>
<tr>
<td></td>
<td>( c = 1/5 )</td>
<td>( c = 1 )</td>
<td>( c = 1/10 )</td>
</tr>
<tr>
<td>Essential ( \delta_{\text{asp}} ) (%)</td>
<td>99.9</td>
<td>100.0</td>
<td>100.0</td>
</tr>
<tr>
<td>110</td>
<td>93.8</td>
<td>92.8</td>
<td>92.5</td>
</tr>
<tr>
<td>120</td>
<td>88.4</td>
<td>83.6</td>
<td>81.7</td>
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<tr>
<td>130</td>
<td>83.8</td>
<td>73.4</td>
<td>70.8</td>
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<tr>
<td>140</td>
<td>79.7</td>
<td>63.8</td>
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</tr>
<tr>
<td>150</td>
<td>75.7</td>
<td>55.4</td>
<td>50.9</td>
</tr>
</tbody>
</table>
7. Numerical Analysis

Panel B: Probabilities evaluated in January 1996 (in %).

<table>
<thead>
<tr>
<th>c = 0</th>
<th>c = 1/50</th>
<th>c = 1/10</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Optimal</td>
<td>S1</td>
</tr>
<tr>
<td>Essential</td>
<td>100.0</td>
<td>99.8</td>
</tr>
<tr>
<td>δ_{asp} (%)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>110</td>
<td>98.9</td>
<td>89.0</td>
</tr>
<tr>
<td>120</td>
<td>96.8</td>
<td>84.0</td>
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<td>90.5</td>
<td>74.9</td>
</tr>
<tr>
<td>c = 1/5</td>
<td>S1</td>
<td>S2</td>
</tr>
<tr>
<td>Essential</td>
<td>99.90</td>
<td>100.0</td>
</tr>
<tr>
<td>δ_{asp} (%)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>110</td>
<td>92.7</td>
<td>92.9</td>
</tr>
<tr>
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<tr>
<td>140</td>
<td>82.9</td>
<td>72.7</td>
</tr>
<tr>
<td>150</td>
<td>80.4</td>
<td>65.6</td>
</tr>
</tbody>
</table>

The investor is a male aged 45 in January 2016 or January 1996 and retiring at 65, in 2036 or 2016. He has an initial capital $W_0$ and plans to make an additional contribution $y = cW_0$ every year. The aspirational goal is a multiple $\delta_{asp}$ of the affordable income at date 0, and the essential goal is to protect 80% of this level at all times. The funded ratio with the probability-maximising strategy is independent from $c$. Strategy S1 uses a customised fund with a simple floor; S2 uses the customised (1, c) fund; strategy S3 is a buy-and-hold combination of the (1,0) and (1,1) funds; strategy S4 uses only the (1,0) fund. For $c = 0$, strategies S2 to S4 are equivalent by construction. A stop-gain decision is made at the first date the funded ratio hits the aspirational level.

After hitting a given aspirational level, however, if it exists, this effect has a low impact, since the four suboptimal strategies allow the essential goal to be secured with probability 1 for each investor. For S1 and S2, this is a numerical confirmation of the theoretical properties of the highly customised and the (1, c) funds. For S3, this follows from Proposition 2, since each investor satisfies $W_{00} = cy_i \leq y_i = D_{00}$. For S4, it is in principle possible for the essential goal not to be secured. Indeed, the (1,0) fund is designed to secure 80% of the purchasing power of the initial contribution alone, but Table 3 shows that it also secures 80% of a stream that consists of $1 at the initial date and $c every year. It should be stressed that this result is obtained through simulations for a specific stochastic model and a specific set of parameter values, but is not a theoretical property of (1,0) funds, and there is no guarantee that it will hold in other settings. A mechanical explanation for why an S4 strategy can secure the essential goal while it has not been designed to do so is as follows. With the relative maximum drawdown floor, the (1,0) fund satisfies, for all dates $u$ and $t \geq u$:

$$X_t \geq \delta_{ess} \frac{X_{tu}}{\beta_u} \beta_t.$$  

Together with (24), this implies that:

$$\frac{W_t + y\alpha_t}{\beta_t} \geq \delta_{ess} \left( \frac{W_0}{\beta_0} + \sum_{u \leq t, u \leq u} \frac{y}{\beta_u} \right) + y \frac{\alpha_t}{\beta_t}.$$  

Let $\nu_{1t}$ be the right-hand side. We have $\nu_{1t} \geq \delta_{ess} r_{\text{max},0}$ if, and only if:
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\[
\delta_{\text{ess}} \sum_{u \in \mathcal{U}_t} \frac{1}{\beta_u} + \frac{\alpha_t}{\beta_t} \geq \delta_{\text{ess}} \frac{\alpha_0}{\beta_0}. \tag{25}
\]

Note that this inequality does not involve the contribution levels, so that the property \( \nu_{1t} \geq \delta_{\text{ess}} r_{\text{max},i,0} \) holds either for all values of \( c \) or for none. Numerically, we find that (25) holds at each date for each scenario with both parameter values of 1996 and 2016. This explains why \( r_{\text{max},i,t} \geq \delta_{\text{ess}} r_{\text{max},i,0} \) at each date with probability 1, so the essential goal is secured.

7.4.3 Probabilities of Reaching Aspirational Goals

We now examine the success probabilities for aspirational goals in Table 3. First, it appears that the ranking of strategies is relatively stable across aspirational goals: the probability-maximising one is far ahead, with probabilities ranging between 90% and 99%. S1 and S2 rank second and third, with S1 being in general the better. S3 ranks third, and S4 yields the lowest probabilities. The cost of implementing the suboptimal but highly customised S1 strategy instead of the optimal one is substantial, particularly for high aspirational levels (140% and 150%) where the decrease in probability exceeds 15%. It should be recalled, however, that the optimal strategy is not implementable in practice if leverage constraints are imposed.

By using the maximum relative drawdown floor to ensure scalability with respect to the entry point (strategy S2), one further reduces, as expected, the probabilities of reaching aspirational goal levels, and it is again for the largest levels of funded ratio (140% and 150%) that the loss is highest. The opportunity cost of limited customisation from S2 to S3 appears to be smaller, with a loss in probability comprised between 0% and 5% in absolute terms. In fact, with the interest rate parameters of January 1996, it happens for some aspirational goals and some contribution values that the combination strategy dominates the customised one, but the difference in probabilities remains at less than 2%. In contrast, suppressing the (1,1) fund to keep only the (1,0) one implies a more significant loss in probability, especially for large goals (\( \delta_{\text{asp}} \geq 130\% \)) and investors with a large ratio \( y/W_0 \), as expected. The difference in probability with respect to the combination strategy easily exceeds 10%, and even 20% in the high interest rate environment of 1996. On the other hand, investors with a low \( y/W_0 \) (1/10 or 1/50) are less affected by the absence of the (1,1) fund.

Figure 2 shows the distributions of the terminal funded ratio achieved respectively with the (1,0) fund and with the highly customised fund when \( m = 3 \). It is seen that the latter distribution has a fatter right tail, but this effect is largely dampened by the stop-gain rule, which makes the investor exit the fund once the funded ratio hits the 130% threshold.
7. Numerical Analysis

Table 4: Comparison of (1,0) fund and heuristic strategies.

Panel A: Probabilities evaluated in January 2016 (in %).

<table>
<thead>
<tr>
<th>c = 0</th>
<th>S4</th>
<th>H1</th>
<th>H2</th>
<th>H3</th>
<th>H4</th>
<th>H5</th>
<th>H6</th>
<th>H7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Essential</td>
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<td>56.3</td>
<td>72.5</td>
<td>49.7</td>
<td>81.5</td>
<td>100.0</td>
<td>65.2</td>
<td>68.5</td>
</tr>
<tr>
<td>δasp (%)</td>
<td>110</td>
<td>91.8</td>
<td>85.1</td>
<td>83.7</td>
<td>44.8</td>
<td>88.1</td>
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<td>84.0</td>
</tr>
<tr>
<td></td>
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<td>83.7</td>
<td>0.0</td>
<td>80.6</td>
</tr>
<tr>
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<td>130</td>
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<td>79.0</td>
<td>0.0</td>
<td>76.7</td>
</tr>
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<td>140</td>
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<td>78.2</td>
<td>71.0</td>
<td>24.4</td>
<td>74.3</td>
<td>0.0</td>
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<tr>
<td></td>
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<td>75.8</td>
<td>66.7</td>
<td>19.5</td>
<td>69.5</td>
<td>0.0</td>
<td>69.7</td>
</tr>
</tbody>
</table>

Panel B: Probabilities evaluated in January 1996 (in %).

<table>
<thead>
<tr>
<th>c = 0</th>
<th>S4</th>
<th>H1</th>
<th>H2</th>
<th>H3</th>
<th>H4</th>
<th>H5</th>
<th>H6</th>
<th>H7</th>
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<tbody>
<tr>
<td>Essential</td>
<td>100.0</td>
<td>73.0</td>
<td>83.6</td>
<td>68.9</td>
<td>91.2</td>
<td>100.0</td>
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<td>89.0</td>
</tr>
<tr>
<td>δasp (%)</td>
<td>110</td>
<td>79.8</td>
<td>88.0</td>
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<td>59.9</td>
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<td>65.7</td>
<td>25.9</td>
<td>59.2</td>
<td>0.0</td>
<td>67.7</td>
</tr>
</tbody>
</table>

The investor is a male aged 45 in January 2016 or January 1996 and retiring at 65, in 2036 or 2016. He has an initial capital $W_0$ and plans to make an additional contribution $y = cW_0$ every year. The aspirational goal is a multiple $\delta_{asp}$ of the affordable income at date 0, and the essential goal is to protect 80% of this level at all times. With strategy S4, all contributions are invested in a (1,0) fund. H1 is 100% in the PSP; H2 is a fixed-mix 50% PSP and 50% bond index; H3 is 100% in the bond index; H4 is 50% PSP and 50% annuity; H5 is 100% in the annuity. H6 and H7 are deterministic target-date funds invested in the PSP and the bond index (H6) or in the PSP and the annuity (H7). In strategy S4 only, a stop-gain decision is made at the first date the funded ratio hits the aspirational level.
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In interpreting the results, it should be remembered that implementation concerns are not taken into account in the success probabilities and should be weighted against the advantages in terms of chances to reach goals. For instance, the optimal policy is not subject to any leverage constraint, and the optimal PSP allocation \( \phi_t \) in Proposition 1 can exceed 100% when \( \delta_{\text{asp}} \) is large.

7.5 Comparison with Other Heuristic Strategies

Most currently available retirement products can be categorised as “balanced funds”, which maintain a constant allocation across their building blocks, as “target date funds” where the allocation is a function of age only, and as annuities. In this subsection, we evaluate the ability of such strategies to meet investors’ goals. We compare them to strategy S4, which invests all contributions in a (1,0) fund. As shown in the previous subsection, it is the fund that implies the lowest probabilities of reaching aspirational goals, so S4 can be regarded as a conservative benchmark. For the alternative “heuristic” strategies, we consider the following options:

- H1 is entirely invested in the PSP;
- H2 is a balanced fund that maintains a constant allocation of 50% to the PSP and 50% to a bond index, modelled as:
  \[
  \frac{dB_t}{B_t} = \left[ r_t + \sigma_B \lambda_B \right] dt + \sigma_B \, dz_{B_t},
  \]
  with \( \sigma_B = 8.3\% \) and \( \lambda_B = 0.39 \) (long-term values from Dimson, Marsh, and Staunton (2008)). The bond index has lower volatility and lower expected return than the stock index that composes the PSP;
- H3 is entirely invested in the bond index;
- H4 is a fixed-mix portfolio with a 50% PSP weight and the remaining 50% invested in the annuity-replicating portfolio (GHP);
- H5 is entirely invested in the GHP;
- H6 is a target-date fund invested in the PSP and the bond index. The initial PSP weight is 90%, and it decreases linearly each quarter until 30% on the last rebalancing date;
- H7 is also a target-date fund, with the same allocation rule, but the bond index is replaced by the annuity-replicating portfolio.

The difference between H2–H3–H6 and H4–H5–H7 is that the bond index is replaced by the annuity. By doing so, one makes use of the proper hedging portfolio. Indeed, unlike the annuity-replicating portfolio, the duration of the bond index does not match that of the annuity, implying that the achieved level of affordable income at retirement date is subject to interest rate risk. Table 4 contains the probabilities of reaching the essential goal and various aspirational goals. To save space, we only report the results for the two extreme values of \( y/W_0 \), namely 0 and 1. We still show them with 1996 and 2016 parameters in order to check the robustness with respect to the interest rate level.

Apart from H5, none of the heuristic strategies reliably secures 80% of the initially affordable income at all dates. The shortfall probability is smallest for H4 and largest for H3. In the latter case, it can reach 50.3% (in 2016 for \( c = 0 \)). H5 is an exception: indeed, for \( c = 0 \), it guarantees a constant ratio \( W/\beta \), hence a constant replacement income. For \( c = 1 \), H5 generates the wealth

\[
W_t = \frac{W_0}{\beta_0} + \sum_{u \leq t} \frac{y_u}{\beta_u} \beta_t
\]
7. Numerical Analysis

so the affordable income \( r_{i, t} = \frac{(W_t + y\alpha_t)}{\beta_t} \) is not constant. But Equation (25), which numerically holds at all dates with probability 1, implies that \( r_{i, t} \geq \delta_{est} r_{i, 0} \) so the essential goal is attained. More generally, as far as the essential goal is concerned, the use of the annuity-replicating portfolio instead of the bond index in a balanced or target-date fund significantly improves the situation: H4 and H7 yield higher probabilities than H3 and H6 respectively. This shows that a proper hedging of the risk factors that affect annuity prices has advantages. But hedging alone is not sufficient, and insurance through dynamic management of the risk budget is needed to protect a minimum level of purchasing power.

With respect to the aspirational goal, most heuristic strategies perform rather well since they do have strong upside potential. Unsurprisingly, it is the one invested in the PSP alone that gives the best success probabilities, which are close to 80% for the 130% funded ratio level, and still greater than 65% for the more ambitious target of 150%. The balanced funds H2 and H4 and the target date funds H6 and H7 also have probabilities greater than 50% for almost all aspirational goals. In contrast, the conservative strategies H3 and H5 imply low, or even zero, probabilities. Recall that for \( c = 0 \), the funded ratio with H5 is constant and equal to 100%, so the probabilities of moving to 110% or higher are zero.

Overall, it appears that balanced funds and target date funds do not correctly account for the need to secure the essential goal when generating upside. Conversely, a full protection of the value of affordable income is achieved by purchasing the annuity-replicating portfolio (H5), but the individual has to give up any hope of moving beyond this level. Strategies that use equities, either alone or with a safe building block, have much more growth potential, but the essential goal is left seriously at risk.

7.6 Positive Effects of Resets

A floor reset mechanism is implemented in (1, 0) funds, where the floor is taken to be:

\[
F_t = \delta_t \left( \max_{s \leq t} \frac{X_s}{\beta_s} \right) \beta_t,
\]

and the coefficient \( \delta_t \) is 85% at time 0 and it is decreased whenever \( F_t/X_t \) exceeds 95%, to the extent that \( \delta_t \geq 80\% \) after the reset. A new fund is launched when \( F_t/X_t \) is still greater than 95% after a reset. This is done to ensure that the new dollars invested by existing clients or new clients benefit from substantial upside potential. On the other hand, launching a new fund implies additional management and distribution costs, so that overly frequent fund creations can be regarded as a disadvantage with respect to the scalability requirement inherent in mass customisation. The reset mechanism is intended to delay these launches of new funds by decreasing the floor when it is too high with respect to the 95% threshold, while using the 85% as a first line of defence. In what follows, we compare the frequency of new fund launches with and without the reset rule.

To this end, we simulate the value of (1,0) funds launched in January 1996 or 2016 for an individual retiring at the age of 65 and currently aged 35, 45 or 55, with funds that have initial horizons of 30, 20 and
7. Numerical Analysis

Table 5: Distribution of first time to launch a new (1,0) fund.


<table>
<thead>
<tr>
<th></th>
<th>35 years</th>
<th></th>
<th>45 years</th>
<th></th>
<th>55 years</th>
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<tr>
<td></td>
<td>R</td>
<td>NR</td>
<td>R</td>
<td>NR</td>
<td>R</td>
<td>NR</td>
</tr>
<tr>
<td>Average time (years)</td>
<td>11.2</td>
<td>8.1</td>
<td>9.5</td>
<td>7.2</td>
<td>6.1</td>
<td>5.0</td>
</tr>
<tr>
<td>Probability (%)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>≤ 1</td>
<td>0.1</td>
<td>1.6</td>
<td>0.1</td>
<td>1.9</td>
<td>0.1</td>
<td>1.9</td>
</tr>
<tr>
<td>≤ 2</td>
<td>1.7</td>
<td>9.6</td>
<td>1.8</td>
<td>9.8</td>
<td>1.8</td>
<td>9.7</td>
</tr>
<tr>
<td>≤ 5</td>
<td>17.4</td>
<td>38.6</td>
<td>17.6</td>
<td>39.1</td>
<td>17</td>
<td>37.5</td>
</tr>
<tr>
<td>≤ 10</td>
<td>49.4</td>
<td>70.1</td>
<td>49.5</td>
<td>70.4</td>
<td>47.8</td>
<td>69.2</td>
</tr>
</tbody>
</table>

Panel B: Funds launched in January 1996.

<table>
<thead>
<tr>
<th></th>
<th>35 years</th>
<th></th>
<th>45 years</th>
<th></th>
<th>55 years</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>R</td>
<td>NR</td>
<td>R</td>
<td>NR</td>
<td>R</td>
<td>NR</td>
</tr>
<tr>
<td>Average time (years)</td>
<td>11.8</td>
<td>8.4</td>
<td>9.6</td>
<td>7.2</td>
<td>6.0</td>
<td>4.9</td>
</tr>
<tr>
<td>Probability (%)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>≤ 1</td>
<td>0.5</td>
<td>4.7</td>
<td>0.4</td>
<td>4.1</td>
<td>0.2</td>
<td>2.9</td>
</tr>
<tr>
<td>≤ 2</td>
<td>3.2</td>
<td>13.2</td>
<td>2.9</td>
<td>12.6</td>
<td>2.5</td>
<td>11.1</td>
</tr>
<tr>
<td>≤ 5</td>
<td>17.7</td>
<td>38.3</td>
<td>17.8</td>
<td>38.2</td>
<td>16.6</td>
<td>36.6</td>
</tr>
<tr>
<td>≤ 10</td>
<td>43.1</td>
<td>65.5</td>
<td>43.9</td>
<td>65.2</td>
<td>42.8</td>
<td>64.8</td>
</tr>
</tbody>
</table>

The (1,0) funds in this table are intended for male investors aged 35, 45 or 55 years at the fund inception date and planning to retire at the age of 65. They are launched in January 1996 or January 2016. Those marked as "R" are subject to floor resets: the coefficient \( \delta_t \) starts at 85% at inception, and when the floor value exceeds 95% of the fund value, it is decreased so as to bring the floor-to-value ratio back to its inception level, unless this operation results in a \( \delta_t \) less than 80%. A new fund is launched on the first date where the ratio is greater than 95% even after the reset. The funds marked as "NR" have a constant coefficient \( \delta_t \) equal to 80% and have no floor resets. For each fund, the table gives the average time until the launch of a new fund and the probability for the fund creation to take place within the first 1, 2, 5 or 10 years.

10 years. For each horizon, we simulate one (1,0) fund according to the base case rebalancing rule, which involves floor resets, and another one in which \( \delta_0 \) is 80% and is thus never decreased. The two situations are respectively referred to as R (resets) and NR (no resets) in Table 5, where we report statistics on the simulated distribution of the time to create the first new fund. By comparing Panels A and B, it is seen that the interest rate parameters have a small to moderate impact on the distribution. The average time is almost identical in both contexts, and the probabilities of having to create a new fund within a given time frame (1, 2, 5 or 10 years) differ by 6% at most. Furthermore, while the average time decreases as the accumulation period shortens, the probability of having to create a fund within 1, 2, 5 or 10 years is almost unchanged. This indicates that the increase in the average time for longer accumulation periods is due to late fund creations, occurring after 10 years.

It clearly appears that the reset process delays the creation of a new fund, whichever indicator is considered. In the case of the 45-year individual, the accumulation period is 20 years long, and the first fund is created after 9.5 years on average with resets, instead of 7.2 years without resets. Similarly, there is a probability of 17.6% only to start a new fund within the first 5 years in the
7. Numerical Analysis

Figure 3: Probabilities of launching at least N (1,0) funds within accumulation period.

(a) 35 years.       (b) 45 years.       (c) 55 years.

A (1,0) fund is launched in January 2016 for male investors currently aged 35, 45 or 55 and retiring at the age of 65, thus in 2046, 2036 or 2026. Funds marked as “R” are subject to floor resets when the floor exceeds 95% of the current fund value: in this event, the coefficient $\delta_t$ starts at 85% and is decreased in the limit of 80%. Funds marked as “NR” have a constant coefficient $\delta_t$ equal to 80%, with no resets. A new fund is launched whenever the floor-to-value ratio is greater than 95% after the reset. The same rules apply to the new fund and so on. The curves represent the probabilities of launching N funds at least within the accumulation period.

first case, a figure that grows to 39.1% in the second case. Similar observations can be made for the other investors.

The new fund can itself become quasi-sterilised in the sense of having a floor-to-value ratio greater than 95% and be replaced by a new one if resets are not possible. The process continues until the cohort’s retirement date. All funds have the same maturity date but staggered inception dates, and the total number of funds created within the accumulation period depends on the scenario. To obtain the distribution of the total number of fund launches, we work scenario by scenario and we simulate the successive funds in each state of the world. Figure 3 summarises the results on the number of creations of new (1,0) funds within the period (including the first fund). The distributions are simulated conditional on 2016 parameters, but very similar pictures, which we do not report here for brevity, have been obtained with those of 1996. The results we obtain are consistent with the results reported in Table 5 in the sense that the number of funds to be created tends to be greater when no resets are applied. For instance, the probability of having to launch at least
7. Numerical Analysis

Table 6: Impact of resets on success probabilities.
Panel A: (1,0) funds launched in January 2016.

<table>
<thead>
<tr>
<th>δ_{asp} (%)</th>
<th>35 years</th>
<th>45 years</th>
<th>55 years</th>
</tr>
</thead>
<tbody>
<tr>
<td>110</td>
<td>R: 94.6</td>
<td>NR: 94.4</td>
<td>R: 91.8</td>
</tr>
<tr>
<td>120</td>
<td>R: 87.7</td>
<td>NR: 86.8</td>
<td>R: 82.8</td>
</tr>
<tr>
<td>130</td>
<td>R: 80.4</td>
<td>NR: 79.4</td>
<td>R: 74.1</td>
</tr>
<tr>
<td>140</td>
<td>R: 73.7</td>
<td>NR: 72.5</td>
<td>R: 65.2</td>
</tr>
<tr>
<td>150</td>
<td>R: 67.6</td>
<td>NR: 66.4</td>
<td>R: 57.7</td>
</tr>
</tbody>
</table>

Panel B: (1,0) funds launched in January 1996.

<table>
<thead>
<tr>
<th>δ_{asp} (%)</th>
<th>35 years</th>
<th>45 years</th>
<th>55 years</th>
</tr>
</thead>
<tbody>
<tr>
<td>110</td>
<td>R: 93.1</td>
<td>NR: 90.6</td>
<td>R: 91.2</td>
</tr>
<tr>
<td>120</td>
<td>R: 88.6</td>
<td>NR: 85.3</td>
<td>R: 84.4</td>
</tr>
<tr>
<td>130</td>
<td>R: 83.7</td>
<td>NR: 80.1</td>
<td>R: 77.6</td>
</tr>
<tr>
<td>140</td>
<td>R: 78.6</td>
<td>NR: 75.3</td>
<td>R: 71.0</td>
</tr>
<tr>
<td>150</td>
<td>R: 74.3</td>
<td>NR: 70.9</td>
<td>R: 65.0</td>
</tr>
</tbody>
</table>

The investor is a male who starts to accumulate money at the age of 35, 45 or 55 in January 2016 or 1996 and plans to retire at age 65. He invests in a (1,0) fund, in which he makes a first contribution at arrival date and brings no additional money thereafter. The aspirational goal is to reach a multiple δ_{asp} of the initially affordable income. Each (1,0) fund is launched at the investor’s arrival date and matures at the retirement date. For each combination of the two dates, there are two possible funds: one in which the floor can be reset to a lower value when it reaches 95% or more of the fund value (R), another where no resets take place (NR). The success probability is the probability of reaching the aspirational level of income at least once during the accumulation phase.

As a conclusion, to delay the creation of a new fund, it is more efficient to start from a coefficient δ_{t} greater than the essential level and to decrease it when needed than to start from the essential level and give up the possibility of resets. This leads to ask whether one could further increase the time to create a new fund by increasing δ_{0} above 85%. Of course, the gain is not unlimited, for taking δ_{0} = 95% would lead to γ_{0} = 95%, but any reset would be without effect because γ_{0} = δ_{0} by construction. Thus, a new fund would have to be started immediately. Hence, there may exist an optimal value for δ_{0} that lies between 85% and 95% and maximises the expected time to create a fund, and estimating this optimum may be a valuable task in implementation.

So as to better understand the trade-offs involved in not utilising the full risk budget when the fund is launched, we report in Table 6 the probabilities of reaching aspirational goals with and without the reset mechanism in the case of a (1,0) fund. For most combinations of an initial investor’s age and aspirational level of funded ratio, the probabilities are actually lower without the resets, which suggests that the benefits of having the possibility to decrease the floor when needed so as to regenerate a risk budget outweigh the opportunity cost of starting with a less aggressive strategy. It is only for the shortest accumulation periods (investors who start saving at age 55 and aiming to retire at 65)
and the most ambitious goals (above 130%) that there would be benefits in giving up the resets. But whichever probability is higher, the difference between the reset and the no-reset cases is very limited, with the difference in probabilities never exceeding 5%. Hence, the decision to reset or not is almost irrelevant from the perspective of the initial investor.

### 7.7 The Perspective of New Investors

We have focused so far on the case of investors who enter the fund at inception date, but (1, c) funds have been designed in order to be scalable, implying that they are intended to protect the essential goal and to allow the aspirational goals to be reached with a positive probability regardless of the entry date. For an individual entering at date $s$, the funded ratio at any date $t$ between $s$ and $T$ is:

$$R_{i,t} = \frac{r_{i, \text{max}, i, t}}{r_{i, \text{max}, i, s}}.$$ 

Consider the case where no intermediate contributions take place $(y_i)$ and the investment support is a $(1,0)$ fund. We have $r_{i, \text{max}, i, t} = W_{i,t} \beta_t$, where $t$ is the annuity price for the fund’s cohort, and $W_{i,t} = (W_{i,t} X_s) X_t$. Hence:

$$R_{i,t} = \frac{R_t}{R_s},$$

where

$$R_t = \frac{X_t}{\beta_t} \times \frac{\beta_s}{X_s} \quad (26)$$

is the funded ratio of the “initial investor” (i.e. the one who enters the fund at date 0). Hence, the funded ratio of new investors can be straightforwardly obtained from the fund and annuity values.

New customers can also take advantage of the creation of new funds and invest in those that have been launched after the initial one has become quasi-sterilised. This choice does not substantially modify the computation of $R_{i,t}$ since the only change is that $X$ in (26) must be interpreted as the value of the most recent $(1,0)$ fund available at date $s$ as opposed to the value of the initial fund. On the other hand, the simulation is more complicated, because launch dates depend on the scenario. Thus, we have to isolate each scenario, simulate the performance of the successive funds along this path (as we did to plot Figure 3), compute the funded ratios by Equation (26) and check for each entry date and each scenario whether the aspirational goal is reached.

Results are summarised in Figure 4. For readability, we focus on a single aspirational goal, which corresponds to 130% of the affordable income of the entry date. The initial fund starts in January 2016 and matures in 2046, 2036 or 2026. Again, we consider the cases with and without resets. Indeed, the previous paragraph has shown that investors who enter the fund at inception date are almost indifferent to the decision to reset, so we want to see whether this is also true for the new ones. The figure also shows the probabilities for investors who choose the most recent fund.

Whichever support is chosen, the probability decreases as the accumulation period shortens. Early investors take advantage of a longer accumulation period, and those who enter the fund less than 10 years before maturity have less than a 50% chance of reaching the 130% target.
7. Numerical Analysis

An initial (1,0) fund is launched in January 2016 for male investors currently aged 35, 45 or 55 and retiring at 65, thus in 2046, 2036 or 2026. In a fund marked as "R", the floor can be reset if it exceeds 95% of the current fund value, while in a fund marked as "NR", the coefficient $\delta_t$ is constant and cannot be decreased. When the floor-to-value ratio exceeds 95% in a fund "R", a new fund is created to replace the quasi-sterilised one. The same fund creation rule applies to the new fund and so on. New investors can arrive at any date between the initial fund inception (2016) and maturity, and they all retire at the fund maturity date. They can either invest in a fund launched in 2016, or in the latest new fund. The curves represent the probabilities for these investors to have their affordable replacement income multiplied by 130% by the time they retire.

The impact of resets is very similar to what was observed for the initial investor. For almost all entry dates, resets imply slightly greater probabilities, but the gain is small in general, being less than 5%. The fact that the impact is positive shows that resets are effective at regenerating some upside potential in the strategy. The largest benefits from resets are for investors who enter the funds maturing in 2046 or 2036 between 2016 and 2026, and thus have at least 10 years of accumulation ahead.

Nevertheless, individuals who have the same investment horizon but invest in funds with different maturity dates are not treated strictly equally. For instance, those who enter in 2036 and retire in 2046 (Panel (a)) have hardly a 30% chance of reaching the objective, while those who enter in 2016 and retire in 2026 (Panel (c)) have a probability greater than 57%. In general, the probabilities have no reason to be identical because of the expected change in economic conditions between 2016 and 2036. But an important factor is that the
7. Numerical Analysis

2016 investor in Panel (c) has a “brand new” fund while the 2036 one in Panel (a) is offered a fund that has accumulated 20 years of growth in the maximum of $X/\beta$. Resets only partially offset the effect of this growth. To avoid investing in a fund with a high floor, new clients may decide to choose the latest fund, which is an intermediate option between the initial fund and a fund that would be launched exactly on their arrival date. The graphs show that this decision has a more positive impact than the decision to reset. In the previous example, for the investor retiring in 2046 and arriving in 2036, the probability grows from about 30% to 47%. Overall, Figure 4 illustrates the benefits of launching new funds, with an impact on success probabilities that is always positive and is especially large for funds that have an initial maturity of 20 or 30 years. On the other hand, the advantages of new fund creations decrease as one approaches the retirement date, because equity returns are too uncertain over short periods to justify a substantial allocation to equities.
8. Conclusion
The general shift towards defined-contribution pension plans leaves individuals with the responsibility of managing the risks associated with their retirement financing needs. The maximum level of replacement income that can be secured given current resources and planned future contributions is determined by current interest rate and expected longevity levels. In most cases, individuals need upside potential to finance higher levels of consumption streams than what is strictly affordable. Having a minimum level of income in all circumstances constitutes an essential goal, while reaching a non-affordable level is an aspirational goal. While a meaningful retirement investment solution should secure the essential goal and deliver a sufficiently large probability of reaching the aspirational one, existing retirement products such as balanced or deterministic target-date funds imply large probabilities of missing this essential goal. In contrast, annuities allow the replacement income level to be known in advance, but this safety comes at the cost of no upside potential. This is only one amongst many explanations of the “annuity puzzle”, the fact that investors do not voluntarily buy substantial amounts of annuities. As a result, no solution currently exists that satisfactorily combines safety and upside potential in terms of replacement income.

In this paper, we argue that existing financial engineering techniques can be used to solve this dilemma. As a first step, we recognise that protecting a minimum level of replacement income is mathematically equivalent to keeping wealth above a floor, the value of which depends on annuity prices and the present value of future contributions. Such a stochastic floor can itself be secured thanks to a suitable extension of portfolio insurance strategies, dynamically invested in a performance building block and a hedging building block so as to generate the upside potential required to reach target levels of replacement income while protecting minimum levels of replacement income. The second challenge posed by the design of retirement investment funds is scalability with respect to individual characteristics. In a retail money management context, full customisation must be replaced by mass customisation, implying that a small number of funds should be used to accommodate the needs of a large number of individuals. We first show that the objectives of individuals with different contribution levels can be met by providing only two elementary retirement investment funds, respectively called the “(1,0)” and the “(1,1)” funds. Scalability with respect to the entry date in the funds raises yet another difficulty, as individuals who start to accumulate at different dates must be entitled to the same value proposal in terms of the protected fraction of their initially affordable replacement income. We show that this can be achieved by introducing a “relative maximum drawdown floor” in each fund and we also introduce floor reset mechanisms to regenerate the risk budget when the floor reaches an exceedingly large value. When reset possibilities are exhausted, the fund is considered to be (quasi) sterilised and a new one with the same maturity date is launched so as to accommodate new investors.

The numerical study highlights several important points. First, the most customised forms of funds have attractive probabilities.
of reaching aspirational goals, even though these probabilities are lower than with the theoretically optimal strategies. Second, limited customisation has a cost in terms of success probabilities, but for reasonable levels of aspirational goals, say around 130%, they still have a substantial chance of success, around 75% for realistic parameter values. Third, the reset mechanism has a positive effect on upside potential, suggesting that relaxing the floor when the risk budget is useful from this perspective, in addition to significantly reducing the number of funds to launch when older funds have been sterilised. While creating new funds to replace sterilised ones can generate operational costs for the fund manager, this discipline has strong positive effects on the success probabilities for new investors, and our results suggest that a parsimonious number of new launches should be considered since they improve the value proposal for new investors while maintaining the scalability requirement in implementation. This approach to "mass customisation" of investment solutions can be applied to other individual goals, such as securing the financing needed for children's education for example.
8. Conclusion
Appendices
A. Proofs of Propositions

A.1 Proposition 1

A.1.1 Optimal Payoff

Our derivation of the optimal payoff follows the lines of Föllmer and Leukert (1999). We define the event $E_0 = \{ M_t \beta_t \leq h \beta_0 \}$, where $h$ is chosen in such a way that $Q_0^h(E_0) = \frac{1}{\Delta_0^\max - \Delta_0^\min}$. Since $\delta = \delta_{\text{est}}^\gamma r_{\text{max}}^\gamma \beta_1$, this condition is equivalent to:

$$E \left[ \frac{M_T (K - 1) G_T}{(K - 1) G_0} \mathbb{1}_{E_0} \right] = \frac{W_0 + Y_0 - G_0}{(K - 1) G_0},$$

hence to:

$$E \left[ M_T \left( G_T + (K - 1) G_T \mathbb{1}_{E_0} \right) \right] = W_0 + Y_0.$$

Denote with $Z^*$ the random variable within the brackets, $Z^* = G_T + (K - 1) G_T \mathbb{1}_{E_0}$, and $Z_t = E_t \left[ \frac{M_T^t}{M^t_0} Z^* \right]$. By the martingale representation theorem, there exists a process $\psi$ such that

$$M_t Z_t^* = W_0 + Y_0 + \int_0^t \psi_s \, dz_s.$$

Write the dynamics of $Z$ as $dZ_t = Z_t [\mu_{Z,t} \, dt + \sigma_{Z,t}' \, dz_t]$ By Ito's lemma:

$$d[M_t Z_t] = M_t Z_t \left[ (-r_t + \mu_{Z,t} - \sigma_{Z,t}' \lambda_t) \, dt + (\sigma_{Z,t} - \lambda_t)' \, dz_t \right],$$

so that $\mu_{Z,t} = r_t + \sigma_{Z,t}' \lambda_t$ and $\sigma_{Z,t} = \psi_t + \lambda_t$. Define a vector of dollar amounts invested in the risky assets as $\theta_t = Z_t \sigma_t^{-1} \lambda_t - Y_t \omega_{\alpha,t}$. The corresponding wealth process evolves as:

$$dW_t = [r_t W_t + \theta_t' \sigma_t' \lambda_t] \, dt + \theta_t' \sigma_t' \, dz_t + y \mathbb{1}_{t \in c}.$$

On the other hand, the present value of contributions evolves as:

$$dY_t = Y_t \left[ r_t + \omega_{\alpha,t}' \sigma_t' \lambda_t \right] \, dt + Y_t \omega_{\alpha,t}' \sigma_t' \, dz_t - y \mathbb{1}_{t \in c}.$$

Summing up the two quantities, we obtain:

$$d(W_t + Y_t) = [r_t (W_t + Y_t) + Z_t \sigma_{Z,t} \lambda_t] \, dt + Z_t \sigma_{Z,t}' \, dz_t,$$

hence:

$$d \left[ e^{-\int_0^t r_s \, ds} (W_t + Y_t) \right] = Z_t e^{-\int_0^t r_s \, ds} \left[ \sigma_{Z,t}' \lambda_t \, dt + \sigma_{Z,t}' \, dz_t \right]$$

$$= d \left[ e^{-\int_0^t r_s \, ds} Z_t \right].$$
This implies that:

\[ W_t + Y_t = \frac{W_0 + Y_0}{Z_0} Z_t = Z_t, \]

hence that \( W_T = Z^* \). Hence, \( Z^* \) is an attainable payoff with the initial wealth \( W_0 \) and the annual contribution \( y \).

The success region, in which the aspirational goal is reached, is:

\[ \{ Z^* \geq K G_T \} = \{(K - 1) \mathbb{1}_{E_0} G_T \geq (K - 1) G_T \} = E_0, \]

where we have used the fact that \( K = \delta_{\text{asp}} / \delta_{\text{ess}} > 1 \). Thus, the probability of success under \( \mathbb{Q}^\beta \) is:

\[
\mathbb{Q}^\beta (Z^* \geq K G_T) = \mathbb{Q}^\beta (E_0) = \mathbb{E} \left[ \frac{M_T \beta_T}{\beta_0} \mathbb{1}_{E_0} \right] \\
= \mathbb{E} \left[ \frac{M_T (K - 1) G_T}{(K - 1) G_0} \mathbb{1}_{E_0} \right] = \frac{W_0 - G_0}{(K - 1) G_0}.
\]

Consider now any strategy with a terminal value \( W_T \) satisfying \( W_T \geq G_T \) almost surely, and let \( E = \{ W_T \geq K G_T \} \) be its success region. We have:

\[
\mathbb{Q}^\beta (E) = \frac{1}{\beta_0} \mathbb{E} [M_T \beta_T \mathbb{1}_E] = \frac{1}{(K - 1) G_0} \mathbb{E} [M_T (K - 1) G_T \mathbb{1}_E].
\]

But, on \( E \), \( W_T \geq K G_T \). Hence:

\[
\mathbb{Q}^\beta (E) \leq \frac{1}{(K - 1) G_0} \mathbb{E} [(W_T - G_T) \mathbb{1}_E].
\]

Because \( W_T - G_T \geq 0 \) and \( \mathbb{1}_E \leq 1 \), it follows that:

\[
\mathbb{Q}^\beta (E) \leq \frac{1}{(K - 1) G_0} \mathbb{E} [W_T - G_T] = \frac{W_0 + Y_0 - G_0}{(K - 1) G_0} = \mathbb{Q}^\beta (E_0).
\]

Then, Neyman-Pearson lemma implies that \( \mathbb{P}(E) \leq \mathbb{P}(E_0) \), so \( Z^* \) is the probability-maximising payoff subject to the constraint of reaching the essential goal.

### A.1.2 Optimal Strategy and Success Probability

The optimal wealth at date \( t \leq T \) is:

\[
W_t^* = \mathbb{E}_t \left[ \frac{M_T}{M_t} Z^* \right] - Y_t = \mathbb{E}_t \left[ \frac{M_T}{M_t} G_T \right] + (K - 1) \mathbb{E}_t \left[ \frac{M_T}{M_t} G_T \mathbb{1}_{E_0} \right] - Y_t \\
= G_t + (K - 1) G_t \mathbb{Q}_t^\beta (E_0) - Y_t \\
= G_t + (K - 1) G_t \mathbb{Q}_t^\beta (M_T \beta_T \leq h \beta_0) - Y_t,
\]
where $Q_t^\beta$ denotes probability under $Q^\beta$ conditional on $\mathcal{F}_t$. To compute the probability of $E_0$, we write $B_t$ as:

$$
\beta_T = \beta_t \exp \left[ \int_t^T \left( r_s + \sigma_{\beta,s} \lambda_{\beta,s} - \frac{\sigma_{\beta,s}^2}{2} \right) ds + \int_t^T \sigma_{\beta,s} \, dz_s \right],
$$

where $\sigma_{\beta,s}$ is the volatility vector of the annuity. Hence:

$$
M_T \beta_T = M_t \beta_t \exp \left[ -\frac{1}{2} \int_t^T \left[ \lambda_{MSR,s}^2 + \sigma_{\beta,s}^2 - 2 \sigma_{\beta,s} \lambda_{\beta,s} \right] ds 
- \int_t^T \left[ \lambda_s - \sigma_{\beta,s} \right]' \, dz_s \right].
$$

(27)

By Girsanov’s theorem, the process $z_t^\beta = z_t + \int_t^T \left[ \lambda_s - \sigma_{\beta,s} \right] \, ds$ is a Brownian motion under $Q^\beta$. Hence:

$$
M_T \beta_T = M_t \beta_t \exp \left[ \int_t^T \left[ \lambda_{MSR,s}^2 + \sigma_{\beta,s}^2 - 2 \sigma_{\beta,s} \lambda_{\beta,s} \right] ds 
- \int_t^T \left[ \lambda_s - \sigma_{\beta,s} \right]' \, dz_s^\beta \right].
$$

The stochastic integral within the brackets can be rewritten as:

$$
\int_t^T \left[ \lambda_s - \sigma_{\beta,s} \right]' \, dz_s^\beta = \int_t^T u_s \, dB_s,
$$

where the process $B$ is defined by $B_0 = 0$ and $dB_t = \left[ \lambda_s - \sigma_{\beta,s} \right] \, dz_s^\beta$, and $u_s^2 = \lambda_{MSR,s}^2 + \sigma_{\beta,s}^2 - 2 \sigma_{\beta,s} \lambda_{\beta,s}$. If $\lambda_{MSR,0}$, $\sigma_{\beta,0}$, and $\lambda_{\beta,0}$ are deterministic functions of time, then $M_t \beta_t$ is log-normally distributed conditional on $\mathcal{F}_t$ under $Q^\beta$. Hence:

$$
W^*_t = G_t + (K - 1) G_t \mathcal{N} (Z_t) - Y_t,
$$

(28)

where $Z_t = \frac{1}{\eta_t} \left[ \ln \frac{h_{\beta,0}}{M_t \beta_t} - \frac{1}{2} \eta_t^2 \right]$ and $\eta_t = \sqrt{\int_t^T u_s^2 \, ds}$.

By applying Ito’s lemma to the right-hand side, we obtain:

$$
dW^*_t = dG_t + (K - 1) \left[ \mathcal{N} (Z_t) \, dG_t + G_t \eta_t (Z_t) \, dZ_t \right] 
- \, dY_t + (\cdots) \, dt,
$$

where the terms in $dt$ do not matter here. Matching the diffusion terms in both sides of this equation, we have:

$$
W^*_t \sigma_t w^*_t = G_t \sigma_t w_{\beta,t} + (K - 1) \mathcal{N} (Z_t) G_t \sigma_t w_{\beta,t}
+ (K - 1) \frac{G_t \eta_t (Z_t)}{\eta_t} \left[ \lambda_t - \sigma_{\beta,t} \right] - Y_t \sigma_t w_{\alpha,t}.
$$
Hence, the optimal portfolio is given by:
\[ w^*_t = \frac{G_t}{W_t^*} \left[ w_{\beta,t} + (K - 1)\mathcal{N}(Z_t)w_{\beta,t} \right. \\
\left. + (K - 1)\frac{n(Z_t)}{\eta_{t,T}} \left[ \sigma_t^{-1} \lambda_t - w_{\beta,t} \right] \right] - \frac{Y_t}{W_t^*}w_{\alpha,t}. \]

Rearranging terms, we obtain:
\[ w^*_t = \varphi_t \frac{\lambda_{MSR,t}}{\sigma_{MSR,t}} w_{MSR,t} + (1 - \varphi_t) w_{\beta,t} - \frac{Y_t}{W_t^*}w_{\alpha,t}, \]
\[ \varphi_t = \frac{(K - 1)n(Z_t)G_t}{\eta_{t,T}W_t^*}. \]

By (28), the coefficient \( Z_t \) can be written as \( \mathcal{N}^{-1} \left( \frac{W_t^* + Y_t - G_t}{(K - 1)G_t} \right) \). To arrive at the expression for \( w^*_t \) given in the text, it suffices to replace \( G_t, KG_t \) and \( W_t^* \) by the following expressions:
\[ G_t = \delta_{\text{ess}} r_{i_{\text{max},0}} \beta_t, \quad KG_t = \delta_{\text{asp}} r_{i_{\text{max},0}} \beta_t, \]
\[ W_t^* = R_t r_{i_{\text{max},0}} \beta_t - Y_t. \]

For the probability, we use the fact that by (27), \( \log M_t \beta_t \) is normally distributed conditional on \( \mathcal{F}_t \) with mean \( \log M_t \beta_t - \eta_{t,T}^2/2 \) and variance \( \eta_{t,T}^2 \), so that:
\[ \mathbb{P}_t(E_0) = \mathcal{N} \left( \frac{1}{\eta_{t,T}} \left[ \ln \frac{h_{t0}}{M_t \beta_t} + \frac{1}{2} \eta_{t,T}^2 \right] \right) \]
\[ = \mathcal{N} \left( Z_t + \eta_{t,T} \right) \]
\[ = \mathcal{N} \left[ \mathcal{N}^{-1} \left( \frac{W_t^* + Y_t - G_t}{(K - 1)G_t} \right) + \eta_{t,T} \right]. \]

### A.2 Proposition 2

For brevity, we omit the individual index \( i \) from the notation. Let
\[ n^{(1,0)}_t = \frac{W_t - D_t}{X^{(1,0)}_t}, \quad n^{(1,c)}_t = \frac{D_t}{X^{(1,c)}_t} \]
be the numbers of shares of each fund purchased at date \( t \). At the first contribution date, date \( u \), the wealth becomes
\[ W_u = n^{(1,0)}_t X^{(1,0)}_t + n^{(1,c)}_t X^{(1,c)}_t + C_u - n^{(1,c)}_t C \]
\[ = n^{(1,0)}_t X^{(1,0)}_t + n^{(1,c)}_t X^{(1,c)}_t + C_u - y \]
\[ = n^{(1,0)}_u X^{(1,0)}_u + n^{(1,c)}_u X^{(1,c)}_u, \]
where the new number of shares of the (1,0) fund is \( n_{u+1}^{(1,0)} = n_t^{(1,0)} + \frac{C_u - y_i}{X_u^{(1,0)}} \).

By repeating this computation at each end of year \( u+1, u+2, ..., T \), we show by induction on \( v \) that for all \( v = 0, ..., T - u \), and any date \( s \) in \([u + v, u + v + 1)\):

\[
W_s = \left[ n_t^{(1,0)} + \sum_{w=0}^{v} \frac{C_{u+w} - y}{X_{u+w}^{(1,0)}} \right] X_s^{(1,0)} + n_t^{(1,c)} X_s^{(1,c)}.
\]

At retirement date:

\[
W_T = \left[ n_t^{(1,0)} + \sum_{w \in S_u} \frac{C_{u+w} - y}{X_{w}^{(1,0)}} \right] X_T^{(1,0)} + n_t^{(1,c)} X_T^{(1,c)}.
\]

By using the fact that \( C_{u+w} - y_i \geq 0 \), \( X_T^{(1,0)} \geq F_T^{(1,0)} \) and \( X_T^{(1,c)} \geq F_T^{(1,c)} \), we obtain, for all \( s \):

\[
\begin{align*}
ri_{\text{max}, T} & \geq \delta_{\text{ess}} \left[ n_t^{(1,0)} + \sum_{w \in S_u, w \leq s} \frac{C_{u+w} - y}{X_{u+w}^{(1,0)}} \right] \frac{X_s^{(1,0)}}{X_s^{(1,c)}} + \delta_{\text{ess}} n_t^{(1,c)} \frac{X_s^{(1,c)}}{\beta_s} + C \alpha_s \frac{X_s^{(1,c)}}{\beta_s} \\
& = \delta_{\text{ess}} \frac{W_s}{\beta_s} + \delta_{\text{ess}} \frac{D_t}{X_t^{(1,c)}} \frac{X_t^{(1,c)}}{\beta_s} C \alpha_s \\
& = \delta_{\text{ess}} \frac{W_s + y_i \alpha_s}{\beta_s}.
\end{align*}
\]

This holds for any \( s \), hence \( ri_{\text{max}, T} \geq \delta_{\text{ess}} \max_{s \in [t, T]} ri_{\text{max}, s} \).

**A.3 Proposition 3**

At date \( t \), the individual purchases \( n_t^{(1,c)} = W_t / X_t^{(1,c)} \) shares of the \( (1, c) \) fund. At the first end of year after arrival, wealth becomes:

\[
W_u = n_t^{(1,c)} X_u^{(1,c)} + C_u - n_t^{(1,c)} C
\]

\[
= n_t^{(1,c)} X_u^{(1,c)} + \frac{W_t}{D_t} y + \left(1 - \frac{W_t}{D_t}\right) y + C_u - y - \frac{W_t}{X_t^{(1,c)}} C
\]

\[
= n_t^{(1,c)} X_u^{(1,c)} + \left(1 - \frac{W_t}{D_t}\right) y + C_u - y
\]

\[
= n_t^{(1,c)} X_u^{(1,c)} + n_t^{(1,0)} X_u^{(1,0)},
\]

where the number of shares of the \( (1,0) \) fund purchased with the extra contribution is \( n_t^{(1,0)} = \frac{1}{X_u^{(1,0)}} \left( C_u - \frac{W_t}{D_t} y \right) \).
Appendices

This computation can be done at each end of year. By induction on \( v \), we verify that for all \( v = 0, \ldots, T - u \) and all \( s \in [u + v, u + v + 1) \):

\[
W_s = n_t^{(1,c)} X_s^{(1,c)} + \sum_{w=0}^{v} \left( C_w - \frac{W_t}{D_t} y \right) X_w^{(1,0)}\cdot
\]

Given that \( C_u \geq y \geq W_t y / D_t \) for all \( u \), we have, at retirement date, for all \( s \in [t, T] \):

\[
W_T \geq \sum_{u \in \mathbb{N}, u \leq s} \left( C_u - \frac{W_t}{D_t} y \right) \frac{1}{X_u^{(1,0)}} \delta_{\text{ess}} \frac{X_s^{(1,0)}}{\beta_s} \beta_T + \frac{W_t}{X_t^{(1,c)}} \delta_{\text{ess}} X_s^{(1,c)} + C\alpha_s
\]

Hence:

\[
W_T \geq \delta_{\text{ess}} \frac{W_s}{\beta_s} + \delta_{\text{ess}} \frac{W_t}{X_t^{(1,c)}} \frac{C\alpha_s}{\beta_s} = \delta_{\text{ess}} \frac{W_s}{\beta_s} + \delta_{\text{ess}} \frac{W_t y_\alpha_s}{D_t} \beta_s.
\]

A.4 Proposition 4

We let \( \tilde{X}_t = (X_t + C\alpha_t) / \beta_t = \bar{X}_t / \beta_t \) be the total fund value expressed in the annuity numeraire, and \( Q_t = \max_{s \leq t} \tilde{X}_s \) be the running maximum of \( \tilde{X} \). By Ito’s lemma, we have:

\[
\frac{d\tilde{X}_t}{\tilde{X}_t} = \frac{dX_t}{X_t} - \frac{d\langle X_t, \beta_t \rangle}{X_t \beta_t} + \frac{d\langle \beta_t \rangle}{\beta_t^2},
\]

where \( \langle \rangle \) denotes quadratic covariation. Using Equation (17) and grouping terms:

\[
\frac{d\tilde{X}_t}{\tilde{X}_t} = m \left( \frac{X_t - \bar{X}_t}{X_t} \right) \left[ \frac{dS_t}{S_t} - \frac{d\beta_t}{\beta_t} - \frac{d\langle S_t, \beta_t \rangle}{S_t \beta_t} + \frac{d\langle \beta_t \rangle}{\beta_t^2} \right]
\]

where \( \bar{S}_t = S_t / \beta_t \) is the PSP value in the annuity numeraire. In order to simplify equations, we work under the probability measure \( \mathbb{Q}^\beta \), with the Brownian motion \( \mathbb{Z}^\beta \) defined as \( d\mathbb{Z}_t^\beta = d\mathbb{Z}_t + [\mathbf{1}_t - \mathbf{1}_t] \cdot d\mathbb{W}_t \). The dynamics of \( \tilde{X} \) under \( \mathbb{Q}^\beta \) read:

\[
\frac{d\tilde{X}_t}{\tilde{X}_t} = m \left( \frac{X_t - \bar{X}_t}{\beta_t} \right) \left[ \sigma_{S,t} - \sigma_{\beta,t} \right]' d\mathbb{Z}_t^\beta,
\]

hence:

\[
d\tilde{X}_t = m \left( \frac{X_t - \bar{X}_t}{\beta_t} \right) \left[ \sigma_{S,t} - \sigma_{\beta,t} \right]' d\mathbb{Z}_t^\beta
\]

\[
= m \left( \tilde{X}_t - \delta_{\text{ess}} Q_t \right) \left[ \sigma_{S,t} - \sigma_{\beta,t} \right]' d\mathbb{Z}_t^\beta.
\]

(29)
In order to alleviate the notation, we let $\delta = \delta_{\text{ess}}$ and following Elie and Touzi (2008), we introduce the auxiliary process

$$U_t = \left[ \tilde{X}_t - \delta Q_t \right] Q_t^{-\delta}.$$

By Ito's lemma, the dynamics of $U$ are (recall that $Q$ is an increasing process and has thus finite variation):

$$dU_t = Q_t^{-\delta} \left[ d\tilde{X}_t - \delta dQ_t \right] + \left[ \tilde{X}_t - \delta Q_t \right] \frac{\delta}{1 - \delta} Q_t^{-\delta} dQ_t.$$

For the second equality, we use the fact that $(Q_t - \tilde{X}_t) dQ_t = 0$. Replacing $d\tilde{X}_t$ by the right-hand side of (29), we obtain:

$$dU_t = mU_t [\sigma_{\beta,t} - \sigma_{\beta,s}]' \delta z_t^\beta. \quad (30)$$

Since $U_0 = (1 - \delta)\tilde{X}_0 Q_0^{-\delta}$ is positive, it follows that $U_t$ is positive for all $t$. Hence, the fund value satisfies $\tilde{X}_t \geq \delta Q_t$. The fact that the fund secures the essential goal of any member of its cohort is then a consequence of the discussion in Section 4.3.

Moreover, by integrating (30) between dates 0 and $T$ and by using the fact that

$$\frac{S_T}{\beta_T} = \frac{S_t}{\beta_t} \exp \left[ -\frac{1}{2} \int_t^T \left[ \sigma_{S,s}^2 + \sigma_{\beta,s}^2 - 2\sigma_{S,\beta,s} \right] ds + \int_t^T [\sigma_{S,s} - \sigma_{\beta,s}]' \delta z_s^\beta \right],$$

we have:

$$\tilde{X}_T = \delta Q_T + \left( \frac{Q_T}{S_T} \right)^{1-\delta} \left( \tilde{X}_t - \delta Q_t \right) \left( \frac{S_t \beta_t}{S_T \beta_T} \right)^m \exp \left[ m(1 - m) \int_t^T \left[ \sigma_{S,s}^2 + \sigma_{\beta,s}^2 - 2\sigma_{S,\beta,s} \right] ds \right]. \quad (31)$$

### B. Construction of Building Blocks

#### B.1 Performance-Seeking Portfolio

Constructing the MSR portfolio is a straightforward procedure once covariances and expected returns are known. In the absence of inequality constraints on weights, the MSR portfolio is given by (6), and the Sharpe ratio maximisation can be performed numerically if short-sales constraints are added.

The main difficulty is in estimating the parameters, given that the estimation errors can outweigh the benefits of scientific diversification with respect to an equally-weighted
portfolio (see Kan and Zhou (2007) and DeMiguel, Garlappi, and Uppal (2009b)). A variety of statistical techniques have been developed to reduce the errors themselves (see DeMiguel, Garlappi, and Uppal (2009b) for a review) or to mitigate their effects by introducing constraints on weights (see Jagannathan and Ma (2003) and DeMiguel et al. (2009a)).

Another approach to constructing the MSR portfolio is to identify a set of portfolios whose returns work as pricing factors. Indeed, it can be shown (see Martellini and Milhau (2015)) that if there exists a set of pricing factors $f$, then the MSR return is a linear combination of these factors:

$$r_{MSR} = \beta_{MSR} f,$$

where $\beta_{MSR}$ is the vector of factor exposures and there is no idiosyncratic risk. This result justifies the claim that diversification should minimise (ideally, cancel) unrewarded risk. Moreover, the MSR portfolio can be constructed by finding an efficient allocation to factor-replicating portfolios. This implies a reduction in the dimensionality of the estimation problem, since only the covariance matrix of the factors and the factor premia need to be estimated. But a crucial step is the identification of the pricing factors. This has been the focus of a voluminous empirical literature searching for factors that explain the cross section of expected returns. The most famous examples in the equity class are the size and the value factors of Fama and French (1993), completed by the momentum factor of Carhart (1997).

The condition of delivering an efficient allocation to rewarded factors and low specific risk define “well-diversified portfolios”. Several weighting schemes often used in the practice of investment attempt to fulfil these conditions without requiring expected return estimates. Among these is the equally-weighted portfolio, which does not require any estimate, but is not efficient unless all assets are indistinguishable (i.e. have equal expected returns, volatilities and correlations). Another choice is the global minimum variance portfolio, which is the only point on the efficient frontier that does not depend on expected returns. Another popular technique is risk parity, which consists in spreading portfolio volatility equally across constituents (Maillard, Roncalli, and Teiletche, 2010). These equal risk contribution portfolios aim to address a well-known shortcoming of equally-weighted portfolios, which is that equal dollar contributions may result in highly unbalanced contributions to risk (see the example of the 60%-40% stock-bond portfolio in Qian (2005)). Pursuing this idea, some authors have proposed to measure diversification in terms of uncorrelated latent factors, which span the same space of uncertainty as the constituents (see Meucci (2009) and Deguest, Martellini, and Meucci (2013)). In this context, the most diversified portfolio is the “factor risk parity” portfolio, in which all factor contributions to volatility are equal. All these weighting schemes can be applied to individual securities or to factor-replicating portfolios in order to construct factor portfolios that proxy for the unknown MSR portfolio.
Appendices

B.2 Annuity-Replicating Portfolio

Survival probabilities at date \( T \) are obtained by adjusting the mortality table of date 0 for expected changes in longevity. In detail, let \( p(t, T, a, T + j) \) be the probability evaluated at date \( t \) that an individual aged \( a \) at date \( T \) dies between dates \( T + j \) and \( T + j + 1 \). We have:

\[
H(T, T, a, s) = \prod_{j=0}^{s-T-1} [1 - p(T, T, a, T + j)].
\]

By treating survival probabilities as deterministic, we replace \( p(T, T, a, T + j) \) by \( p(t, T, a, T + j) \). A period mortality table at date \( t \) contains the probabilities \( p(t, t, a, t) \) for various ages \( a \), but it is expected that mortality will decrease in the future, so that \( p(t, T, a, T + j) \) is taken lower than \( p(t, t, a, t) \).\(^{10}\) To capture the decreasing trend in mortality, we apply a projected reduction factor in mortality to transform the probabilities at the base date of the mortality table into future probabilities. By taking date 0 as the reference date, we have (see Dickson, Hardy, and Waters (2013) p. 68):

\[
p(t, T, a, T + j) = p(t, t, a, T + j)[1 - \varphi(a)]^{T+j-t},
\]

where \( \varphi(a) \) is the annual mortality reduction factor at age \( a \). Because mortality probabilities are lower with this method than by assuming stationary probabilities, annuity prices are higher.

---

\(^{10}\) A period table gives the mortality probabilities within the reference population in a given year: formally, it provides \( p(t, t, a, t) \) for a range of ages \( a \). The other family of tables is cohort tables, which describe the mortality experience of a group of individuals born in the same year. A cohort table at date \( t \) for the cohort born at \( u \) provides \( p(t, u, 0, u) \) for various \( u \).
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- Indices and benchmarking
- Non-financial risks, regulation and innovations
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- ALM and asset allocation solutions

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