EDHEC–Risk Days
North America 2013

Bringing Research Insights to Institutional Investment Professionals

8–9 October, 2013 — New York
Hedging Long-Term Inflation-Linked Liabilities without Inflation-Linked Instruments

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This research has been supported by Ontario Teachers' Pension Plan in the context of the "Advanced Investment Solutions for Liability Hedging for Inflation Risk" research chair
Outline

- Inflation risk versus liability risk
- Expected inflation risk versus realized inflation risk
- Diversifying versus hedging expected inflation risk
- Inflation risk versus liability risk

- Expected inflation risk versus realized inflation risk

- Diversifying versus hedging expected inflation risk
Inflation Hedging without Inflation-Linked Bonds

- The lack of capacity of inflation-linked (IL) bond markets, and the increased concern over counterparty risk for derivatives-based solutions, leave most investors with the presence of non-hedgeable inflation risk.

- This is a key concern since most investors implicitly or explicitly face IL liabilities, or more generally consumption needs.

- In this context, a variety of traditional asset classes (stocks and bonds) and alternative asset classes (commodities or real estate in particular) have been analyzed in terms of their ability to provide attractive inflation-hedging benefits.

- The results of such empirical investigations have been mixed, with results that are subject to substantial model and parameter uncertainty.
Inflation Hedging with Financial Assets

Estimated correlation between asset classes and realized inflation

- Stocks (S), long-term nominal bonds (B) and commodities (Com) have an upward-slopping term structure of correlation with realized inflation.

- Stocks (S), real estate (RE) and commodities (Com) have a positive correlation with inflation (stronger results could be obtained by increasing granularity (see for example Ang et al. (2012) for equity markets); on the other hand, the short-term correlation between the return on inflation-linked bonds (I) and inflation is close to zero (!?).

(*) Results based on Vasicek model calibrated over the period Q2.1961 – Q3.2011.
Better results could be obtained by increasing granularity (see for example Ang et al. (2012) for equity markets).

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<td>Treasuries</td>
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<td>Sec. Cons. Goods</td>
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<td>Sec. Cons. Services</td>
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<td>Com. Energy</td>
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What justifies the interest in inflation-hedging properties of various asset classes is the intuition that investors with inflation-linked liabilities need to invest in assets that are positively correlated with inflation.

This seemingly straightforward intuition is wrong, or at least severely incomplete (as can be guessed from the virtually zero correlation between return on IL bonds and inflation).

What matters is not the inflation-hedging properties of various asset classes, but instead their liability-hedging properties.
Inflation Hedging versus Interest Rate Hedging

- Liability risk contains interest rate risk(s) in addition to inflation risk(s), so inflation risk hedging and inflation-linked liability risk hedging are two distinct concepts that coincide only at liability maturity.

- For reasonable parameter values, interest rate risk dramatically dominates inflation risk when it comes to the contribution to short-term volatility of the funding ratio for long-term IL liabilities.

- From a mathematical point, this can be explained as follows:
  - Interest rate risk affects liability risk through the impact on the discount factor, an impact that increases with time-horizon.
  - Inflation risk affects the value of the liabilities through an impact on the cash-flows, which is not affected by time-horizon.
  - It is only in the case of short time-to-horizon (say 1 year) that inflation risk becomes relatively substantial within total liability risk.
Short–Term Liability Risk

Short–term correlation between bond portfolios and liabilities (%)

<table>
<thead>
<tr>
<th>Maturity</th>
<th>15 Y</th>
<th>10 Y</th>
<th>5 Y</th>
<th>1 Y</th>
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<tbody>
<tr>
<td>Bond portfolios</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>B (10Y)</td>
<td>99.73</td>
<td>99.54</td>
<td>98.63</td>
<td>75.62</td>
</tr>
<tr>
<td>B, I (10Y)</td>
<td>100.00</td>
<td>100.00</td>
<td>100.00</td>
<td>100.00</td>
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</tbody>
</table>

- The instantaneous correlation between nominal bonds and IL liabilities is close to 100 % for maturities 10Y and 15 Y
- On the other hand, for short liability maturities (1Y), nominal bonds have lower correlation with liabilities – this correlation falls to 0 for vanishing maturities (i.e., at maturity date).

(*) Results based on Vasicek model calibrated over the period Q2.1961 – Q3.2011.
**Short–Term versus Long–Term Liability Risk Hedging**

- There exists a fundamental trade-off/conflict between short-term and long-term IL liability risk hedging:
  - Since real assets (e.g., commodities) have no well-defined (real) interest rate hedging properties, their presence in liability-hedging portfolio would generate high short-term volatility in funding ratio levels, and nominal bonds should be clearly preferred as substitutes for IL bonds.
  - At horizon, however, the correlation between the (constant) payoff of nominal bonds and the (inflation-linked) payoff of inflation-linked liability is zero: nominal bonds have no hedging power for inflation-linked liability payments.

- In practice, the short-term perspective dominates, and most investors only have fixed-income instruments in their liability-hedging portfolios, because of their attractive interest rate hedging properties.
- Inflation risk versus liability risk
- Expected inflation risk versus realized inflation risk
- Diversifying versus hedging expected inflation risk
Realized versus Expected Inflation Risk

- If realized inflation risk is not a serious source of short-term funding ratio volatility for long-term constant maturity IL liabilities, expected inflation risk can be.

- Nominal bonds are exposed to changes in real rates and expected or break-even inflation (\(^\ast\)), while IL bonds and IL liabilities are only exposed to changes in real rates (and realized inflation).

\[
B_t = PV_t(100) \Rightarrow \frac{dB_t}{B_t} = \mu_B dt - D_{\pi} (\tau_B - t) \sigma_{\pi} dz_{t}^\pi - D_{\pi} (\tau_B - t) \sigma_{\pi} dz_{t}^\pi
\]

\[
L_t = PV_t(100 \times \Phi_T) \Rightarrow \frac{dL_t}{L_t} = \mu_L dt - D_{\pi} (\tau_L - t) \sigma_{r} dz_{t}^r + \sigma_\phi dz_{t}^\phi
\]

- The concern would be a strong increase in expected inflation (which would lead to a drop in nominal bond prices), typically in a context where realized inflation is high.

\(^\ast\) Break–even inflation rate = nominal rate – real rate; changes in break–even rates reflect changes in inflation expectations, but also possible changes in inflation risk premium as well as spurious effects related to IL bond market liquidity factors.
We consider four relatively short periods with high increase in expected inflation (from a local minimum to a local maximum).

(*) Expected inflation is estimated using the methodology in Kothari and Shanken (2004, FAJ).
A massive underperformance of nominal bonds with respect to IL liabilities is obtained on these periods.

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<td>Absolute increase in expected inflation (in %) (*)</td>
<td>6.23</td>
<td>6.77</td>
<td>2.84</td>
<td>3.82</td>
</tr>
<tr>
<td>Return on GSCI (in %)</td>
<td>192.93</td>
<td>87.84</td>
<td>36.98</td>
<td>6.03</td>
</tr>
<tr>
<td>Return on 10-year nominal bonds (in %)</td>
<td>−1.84</td>
<td>−8.41</td>
<td>−4.08</td>
<td>−0.61</td>
</tr>
<tr>
<td>Return on 10-year liabilities (in %) (*)</td>
<td>24.49</td>
<td>23.93</td>
<td>4.39</td>
<td>4.13</td>
</tr>
</tbody>
</table>

(*) Expected inflation and liability returns are estimated using the methodology in Kothari and Shanken (2004, FAJ). Actual liability returns are used after 2002.
Inflation risk versus liability risk

Expected inflation risk versus realized inflation risk

Diversifying versus hedging expected inflation risk
Inflation Hedging – Outstanding Questions

Hedging IL liabilities is a key challenge in the absence of IL linked bonds, which are the only (cash) securities that have perfect hedging properties for IL liabilities at all horizons.

In this context, a number of important research questions stand out, with a positive answer already provided for the first one:

- **Q1**: Can we find economic periods/regimes where long positions in nominal bonds, which are the natural substitutes to IL bonds, are unable to provide satisfactory hedging for IL liabilities, because of their inability to hedge against changes in break-even inflation?
- **Q2**: Can we use real assets in liability-hedging portfolios to compensate for the risk of the poor performance of long positions in nominal bonds in case of a jump in break-even inflation?
- **Q3**: Alternatively, can we neutralize expected inflation risk exposure in nominal bonds so as to generate a better match with respect to the inflation-linked liability portfolios?
Managing Expected Inflation Risk: Possible Approaches

- **Diversifying** away expected inflation risk (Q2)
  - Static approach: the strategy here would consist in holding at all times a *static* mix of nominal bonds and real assets – problem here is that the allocation to real assets will be much too low in case of a surge in expected inflation, and much too high in all other market conditions.
  - Dynamic approach: the strategy here would consist in holding at all times a *dynamic* mix of the optimal LHPs under each particular regime; the weights assigned to each LHP are taken to be a function of the filtered probabilities for each regime to prevail looking forward.

- **Hedging** away expected inflation risk (Q3)
  - Match the expected inflation exposure (that is neutralize it) and the real rate exposure with respect to the liabilities.
  - In principle, this can be achieved through a suitably-designed dynamic long-short portfolio strategy in nominal bonds; in practice, we need to analyze whether this approach would work out-of-sample.
A Formal Model

- We use the model of Munk, Sørensen and Vinther (2004) which allows for a stochastic expected inflation process that evolves as:

\[ d\pi_t = \varphi(\bar{\pi} - \pi_t)dt + \sigma_{\pi} d\zeta^{\pi}_t \]

and the price index follows:

\[ \frac{d\Phi_t}{\Phi_t} = \pi_t dt + \sigma_{\Phi} d\zeta^{\Phi}_t \]

- This model accounts for the fact that neither expected inflation risk nor real rate risk is entirely spanned by nominal bonds.

- The ability of nominal bonds to hedge IL liabilities depends on:
  - Magnitude of expected inflation risk (volatility \( \sigma_{\pi} \), speed of mean reversion \( \varphi \)),
  - The magnitude of realized inflation risk (volatility \( \sigma_{\Phi} \)).
Preview of Results

Using a two-state Markov regime switching model, we find that:

- State 1 corresponds to the larger uncertainty on expected and realized inflation, with higher volatilities for both processes; it does not distinguish however between increases and decreases in expected inflation;
- State 2 corresponds to the lower uncertainty: both processes have lower volatilities, although expected inflation has lower speed of mean reversion.

The estimated speed of mean reversion is not lower in regime 1 than in regime 2, but it is very imprecisely estimated, with a standard error that represents at least half the estimate.

Filtered and smoothed probabilities allow one to clearly identify the two inflation regimes.
Estimated Parameters and Probabilities

All parameters are state-dependent.

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<tr>
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<tbody>
<tr>
<td></td>
<td>State 1</td>
<td>State 2</td>
<td></td>
</tr>
<tr>
<td><strong>( \varphi )</strong></td>
<td>0.5239 (0.2682)</td>
<td>0.4168 (0.1705)</td>
<td></td>
</tr>
<tr>
<td><strong>( \pi_\tau )</strong></td>
<td>0.0457 (0.0163)</td>
<td>0.0408 (0.0054)</td>
<td></td>
</tr>
<tr>
<td><strong>( \sigma_\pi )</strong></td>
<td>0.0324 (0.0041)</td>
<td>0.0124 (0.0008)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>State 1</td>
<td>State 2</td>
<td></td>
</tr>
<tr>
<td><strong>( \sigma_\Phi )</strong></td>
<td>0.0181 (0.0021)</td>
<td>0.0064 (0.0005)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>State 1</td>
<td>State 2</td>
<td></td>
</tr>
<tr>
<td><strong>( \rho_{\pi\Phi} )</strong></td>
<td>0.6078 (0.0959)</td>
<td>0.4207 (0.0776)</td>
<td></td>
</tr>
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</table>

Transition matrix.

<table>
<thead>
<tr>
<th>From State 1</th>
<th>From State 2</th>
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<tbody>
<tr>
<td>State 1</td>
<td>0.8650 (0.0612)</td>
</tr>
<tr>
<td>State 2</td>
<td>0.1350 (−)</td>
</tr>
</tbody>
</table>
Dynamic Hedging Strategies

- The presence of regime switches motivates the use of dynamic liability-hedging portfolio strategy, by switching from a 100% nominal bond LHP to a LHP containing some allocation to commodities when it is recognized that the high expected inflation regime has materialized, for some criterion to be defined:

\[
\begin{pmatrix}
\text{Nom. Bonds} \\
\text{Commodities}
\end{pmatrix} = \begin{pmatrix}
1-x \\
x
\end{pmatrix}
\]

- In what follows, we take \( x = 25\%, 50\% \) or 75\%, and the portfolio is then left buy-and-hold until exit from the regime.

- We consider an idealized shifting strategy with no lag, and a strategy with a shift that occurs with a 6 months lag.
### Dynamic Diversification Strategies – No Lag
#### 1972 to 1989

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<td>4.61</td>
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<tr>
<td>Absolute increase in Michigan Survey (in %)</td>
<td>NA</td>
<td>NA</td>
<td>2.4</td>
<td>1.3</td>
</tr>
<tr>
<td>Absolute increase in 20–year break-even rate (in %)</td>
<td>NA</td>
<td>NA</td>
<td>NA</td>
<td>NA</td>
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<tr>
<td>Return on GSCI (in %)</td>
<td>192.93</td>
<td>87.84</td>
<td>26.57</td>
<td>75.74</td>
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<tr>
<td>Return on 20–year liabilities (in %)</td>
<td>163.15</td>
<td>189.8</td>
<td>45.56</td>
<td>171.17</td>
</tr>
<tr>
<td>Return on dynamic strategy with $x = 25%$ and no lag (in %)</td>
<td>27.16</td>
<td>–0.03</td>
<td>–13.53</td>
<td>23.65</td>
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<td>Return on dynamic strategy with $x = 50%$ and no lag (in %)</td>
<td>82.41</td>
<td>29.26</td>
<td>–0.16</td>
<td>41.01</td>
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<td>Return on dynamic strategy with $x = 75%$ and no lag (in %)</td>
<td>137.67</td>
<td>58.55</td>
<td>13.2</td>
<td>58.38</td>
</tr>
<tr>
<td>Return on dynamic strategy with $x = 100%$ and no lag (in %)</td>
<td>192.93</td>
<td>87.84</td>
<td>26.57</td>
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</table>

**Outperformance w.r.t. liabilities.**

**Underperformance w.r.t. liabilities.**

(*) Before April 1999, expected inflation and liability returns are estimated using the methodology in Kothari and Shanken (2004, FAJ).
**Dynamic Diversification Strategies – 6 Months Lag**

1972 to 1989

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<td>45.56</td>
<td>171.17</td>
</tr>
<tr>
<td>Return on dynamic strategy with $x = 25%$ and 6–M lag (in %)</td>
<td>18.83</td>
<td>– 3.78</td>
<td>– 14.94</td>
<td>26.37</td>
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<td>69.41</td>
<td>21.05</td>
<td>– 6.80</td>
<td>34.26</td>
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<td>Return on dynamic strategy with $x = 75%$ and 6–M lag (in %)</td>
<td>119.99</td>
<td>45.87</td>
<td>1.35</td>
<td>42.14</td>
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<tr>
<td>Return on dynamic strategy with $x = 100%$ and 6–M lag (in %)</td>
<td>170.57</td>
<td>70.69</td>
<td>9.49</td>
<td>50.02</td>
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# Dynamic Diversification Strategies – No Lag 1992 to 2009

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<tbody>
<tr>
<td>Absolute increase in expected inflation (in %) (*)</td>
<td>2.85</td>
<td>2.84</td>
<td>4.75</td>
<td>3.82</td>
</tr>
<tr>
<td>Absolute increase in Michigan Survey (in %)</td>
<td>0.4</td>
<td>0.9</td>
<td>2.5</td>
<td>0</td>
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<tr>
<td>Absolute increase in 20–year break-even rate (in %)</td>
<td>NA</td>
<td>NA</td>
<td>0.41</td>
<td>0.44</td>
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<td>– 6.18</td>
<td>36.98</td>
<td>173.19</td>
<td>6.03</td>
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<tr>
<td>Return on 20–year nominal bonds (in %)</td>
<td>27.01</td>
<td>– 0.15</td>
<td>54.18</td>
<td>– 4.35</td>
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<tr>
<td>Return on 20–year liabilities (in %)</td>
<td>124.07</td>
<td>16.27</td>
<td>68.76</td>
<td>4.43</td>
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<tr>
<td>Return on dynamic strategy with (x = 25%) and no lag (in %)</td>
<td>18.71</td>
<td>9.13</td>
<td>83.93</td>
<td>– 1.76</td>
</tr>
<tr>
<td>Return on dynamic strategy with (x = 50%) and no lag (in %)</td>
<td>10.41</td>
<td>18.41</td>
<td>113.68</td>
<td>0.84</td>
</tr>
<tr>
<td>Return on dynamic strategy with (x = 75%) and no lag (in %)</td>
<td>2.11</td>
<td>27.70</td>
<td>143.44</td>
<td>3.44</td>
</tr>
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<td>Return on dynamic strategy with (x = 100%) and no lag (in %)</td>
<td>– 6.18</td>
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**Dynamic Diversification Strategies – 6M Lag 1992 to 2009**

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<td>Return on dynamic strategy with $x = 25%$ and 6–M lag (in %)</td>
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<td>13.83</td>
<td>67.59</td>
<td>NA</td>
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<tr>
<td>Return on dynamic strategy with $x = 50%$ and 6–M lag (in %)</td>
<td>3.36</td>
<td>21.24</td>
<td>88.16</td>
<td>NA</td>
</tr>
<tr>
<td>Return on dynamic strategy with $x = 75%$ and 6–M lag (in %)</td>
<td>– 3.10</td>
<td>28.66</td>
<td>108.73</td>
<td>NA</td>
</tr>
<tr>
<td>Return on dynamic strategy with $x = 100%$ and 6–M lag (in %)</td>
<td>– 9.55</td>
<td>36.07</td>
<td>129.31</td>
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(*) Before April 1999, expected inflation and liability returns are estimated using the methodology in Kothari and Shanken (2004, FAJ).
Expected Inflation Risk: Diversification vs. Insurance

- In the absence of IL bonds, commodities are useful additions to nominal bonds provided that we have a substantial allocation to them when needed, as opposed to a small allocation to them all the time (dynamic vs. static approach).
  - In the end, the problem is not so much about *diversification* (minimization of the volatility of the funding ratio over long periods of time) than it is about *insurance* (getting protection in some very few specific economic conditions).
  - We need to calibrate a parsimonious model to switch to the commodities-dominated LHP (*before it is too late?*)

- Which model to use is unclear: using a formal MRS model would not really help since it may not distinguish between increases and decreases in expected inflation; perhaps a model based on observable state variable would do better?
All the afore-mentioned challenges could be avoided if one could neutralize the exposure to unexpected inflation in nominal bond portfolios, while matching the real rate exposure with respect to the liabilities.

In principle, this can be achieved through a suitably-designed portfolio strategy involving long/short portfolios of nominal bonds (a short position in nominal bonds would generate a profit in case of a large increase in expected inflation, \textit{without the need for timing the regime}).

In practice, we need however to analyze the out-of-sample robustness of the approach in the presence of parameter uncertainty.
Model-Free Bond Prices

- In general, nominal bond prices can be written as function of real yields and break-even expected inflation:

\[
\begin{align*}
B_t^1 &= \exp\left(-(\tilde{r}_{t,T_1} + \pi_{t,T_1})(T_1 - t)\right) \Rightarrow \frac{\partial B_t^1}{\partial \tilde{r}_{t,T_1}} &= \frac{\partial B_t^1}{\partial \pi_{t,T_1}} = -(T_1 - t) B_t^1 \\
B_t^2 &= \exp\left(-(\tilde{r}_{t,T_2} + \pi_{t,T_2})(T_2 - t)\right) \Rightarrow \frac{\partial B_t^2}{\partial \tilde{r}_{t,T_2}} = \frac{\partial B_t^2}{\partial \pi_{t,T_2}} = -(T_2 - t) B_t^2
\end{align*}
\]

- Inflation-indexed bond prices written as function of maturity-dependent real interest rate and CPI level:

\[
I_t = \Phi_t \exp\left(-\tilde{r}_{t,\tau_L} (\tau_L - t)\right) \Rightarrow \frac{\partial I_t}{\partial \tilde{r}_{t,\tau_L}} = -(\tau_L - t) I_t
\]
Estimation with OLS

- Assumption on shifts in real interest rate and break-even expected inflation:
  \[
  d\tilde{\tau}_{t,T_2} = b^{T_2,T_1} d\tilde{\tau}_{t,T_1}; \quad d\tilde{\tau}_{t,L} = b^{L,T_1} d\tilde{\tau}_{t,T_1}
  \]
  \[
  d\pi_{t,T_2} = \beta^{T_2,T_1} d\pi_{t,T_1}
  \]

- Intuition suggests that long-term inflation expectations are less volatile than short-term inflation expectations (i.e., \(\beta(T_2, T_1) < 1\)).

- Nominal and real yields are observed, and break-even expected inflation inferred from observed yields:
  \[
  \begin{aligned}
  \pi_{t,T_1} &= y_{t,T_1} - \tilde{\tau}_{t,T_1} \\
  \pi_{t,T_2} &= y_{t,T_2} - \tilde{\tau}_{t,T_2}
  \end{aligned}
  \]

- Regression equation:
  \[
  \pi_{t,T_2} = \beta_0^{T_2,T_1} + \beta^{T_2,T_1} \pi_{t,T_1} + \epsilon_{\beta^{T_2,T_1}}
  \]
Replicating Strategy

- $L_t$: constant-maturity liability portfolio

- $A_t$: replicating portfolio constituted by two nominal bonds and cash

\[
\frac{dA_t}{A_t} = w_{1t} \frac{dB^1_t}{B^1_t} + w_{2t} \frac{dB^2_t}{B^2_t} + (1 - w_{1t} - w_{2t}) r_t dt
\]

- **Main idea**: matching the exposures to changes in the real interest rate and break-even expected inflation.

- We obtain the following system (to which can potentially be added leverage constraints):

\[
\begin{align*}
    w_{1t}T_1 + w_{2t}T_2b^{T_2,T_1} &= \tau_L b^{\tau_L,T_1} \\
    w_{1t}T_1 + w_{2t}T_2\beta^{T_2,T_1} &= 0
\end{align*}
\]
Long-Short Portfolio

- The solution is a long-short portfolio strategy:

  Expression of weights:

  
  \[ w_{1t} = \frac{1}{T_1(\beta_{T_2,T_1} - b_{T_2,T_1})} \tau_L^{\beta_{T_2,T_1}} \beta_{T_2,T_1} \]

  \[ w_{2t} = -\frac{1}{T_1(\beta_{T_2,T_1} - b_{T_2,T_1})} \tau_L^{\beta_{T_2,T_1}} \]

  Expected sign (\( T_1 < T_2 \)):

  \[ - \]

  \[ + \]

- Expected signs are based on the intuition that:

  \[ \beta(T_2,T_1) < b(T_2,T_1) \]

  Decreasing in \( T_2 \) \( \approx 1 \)

- We use \( T_1 = 5Y, T_2 = 20Y, \tau_L = 10Y \), and we estimate the betas over 2-year rolling windows.
The strategy deviates from liabilities in November 2008, when there was a substantial drop in interest rates (+100 bps).

Accuracy of the strategy could be improved by adding a convexity adjustment. (*)

(*) It can also be potentially improved by seeking to neutralize changes in the shape of the yield curve using a parsimonious model such as the Nelson-Siegel model.
**Constraining Parameters $b(T_2, T_1)$ and $b(\tau_L, T_1)$**

- Another idea is to set $b(T_2, T_1) = b(\tau_L, T_1) = 1$ so as to reduce the number of parameters to estimate and to limit the volatility of the weights.

- This has a negative impact on the long-term performance.

*Strategy with 150% total leverage constraint.*
Forward-Looking Estimates

- Coefficients are not constant over time!

- Idea is thus to generate forward-looking estimates for $b$ and $\beta$ by regressing rolling-window estimates onto a set of predictive variables.
We run the following predictive regression for each coefficient:

$$\beta_{t+1}^{T_2, T_1} = \kappa_0 + \kappa_1' X_t + \epsilon_{t+1}$$

where $X$ is a vector of predictors that contains the current estimated coefficients $\beta(T_2, T_1)$, $b(T_2, T_1)$ and $b(\tau_L, T_1)$, as well as two macro-economic predictors ($2Y$ break-even rate and unemployment rate - Ang and Piazzesi (2003)).
Backtest of Replicating Strategy – 2002 - 2012

- As before, we impose a 150% total leverage constraint.

- The replication is more accurate than with rolling-window estimates for the coefficients.
Main Conclusions

- Liability risk hedging is different from inflation risk hedging:
  - Interest rate risk strongly dominates (realized) inflation risk within short-term liability risk, except at liability maturity.
  - Nominal bonds appear as a good substitute for IL bonds, except in case of a surge in expected inflation.

- This concern can be addressed in two possible ways:
  - Add real assets to provide diversification if and when needed – but they probably belong to the PSP, not LHP.
  - Implement L/S nominal bond allocation strategies so as to hedge away exposure to expected inflation.

- Both methods suffer from a number of shortcomings – when feasible, using IL bonds is the only option that always works.