The State of the Art in Extreme Risk Measurement

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  - Which approach is the best?
  - Toward the integration of Extreme Values
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  - Block Maxima
  - Peak over Threshold
- Conclusion

Introduction

Objective

To show that :

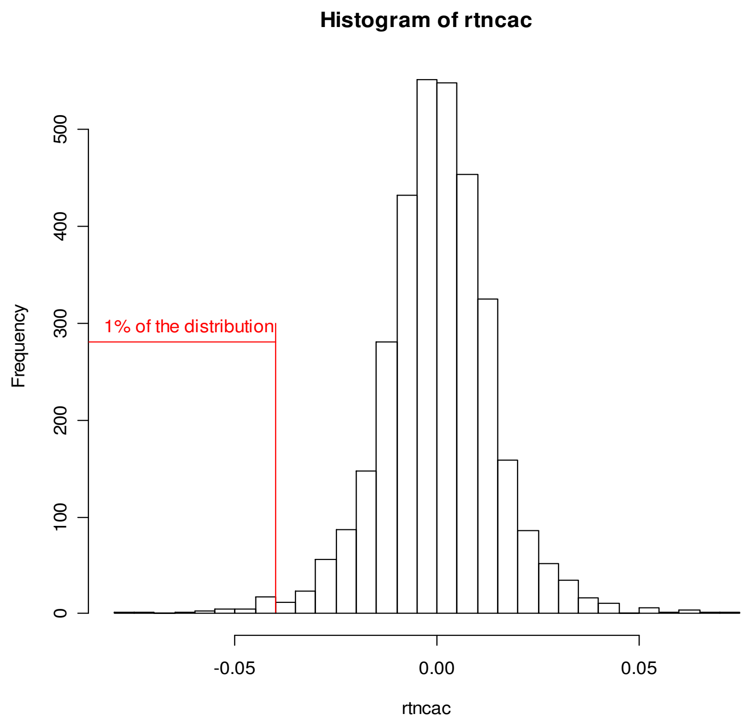
- VaR is not the best risk measure
- Even if we consider measures beyond the VaR we must integrate extreme values explicitly in the modelling
- How to deal with extreme values

Is the Value-at-Risk a panacea?

- Need to assess risk (market, credit, operational, etc.)
- VaR has progressively become a standard
- The concept is very simple and leaves no room for differing interpretations
- It corresponds to "the possible loss that can be sustained for a given period and for a given confidence level"

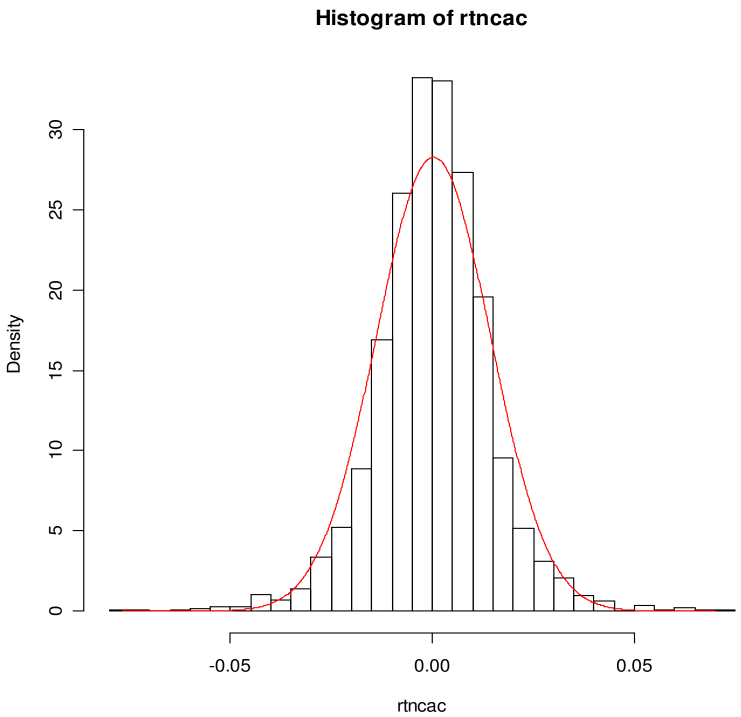
Which approach is the best ?

- 3 main approaches
  - Historic: just apply current weights to a time series of historical asset returns. VaR is simply the α% quantile of this distribution



The first percentile is -0.04 so VaR(1%) is 0.04

– Analytic : we suppose that the variations in the portfolios could be forecasted by a parametric distribution. Most explicitly or implicitly suppose normality (like Riskmetrics approach).

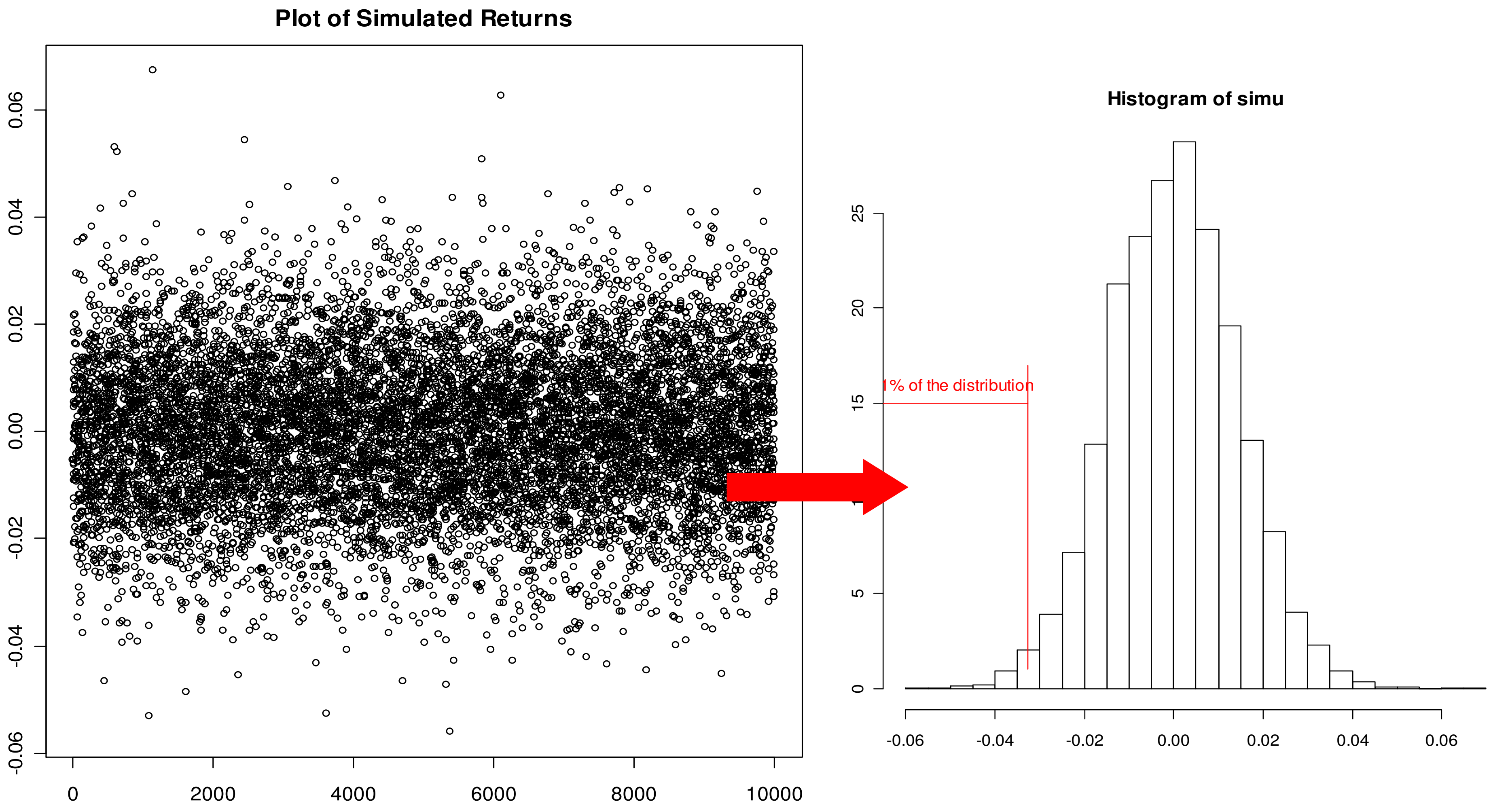


μ=0.0002851

σ=0.014124

VaR(1%) is $\Phi^{-1}(0.01)$*sd(rtncac) +mean(rtncac) so VaR=-0.032573

– Simulation: As in the previous approach, parameters for parametric distributions are estimated from past data. In a second step those parameters are used to generate random values for the risk factors from which we will obtain the value of the portfolio. Then we only have to select the α% quantile.



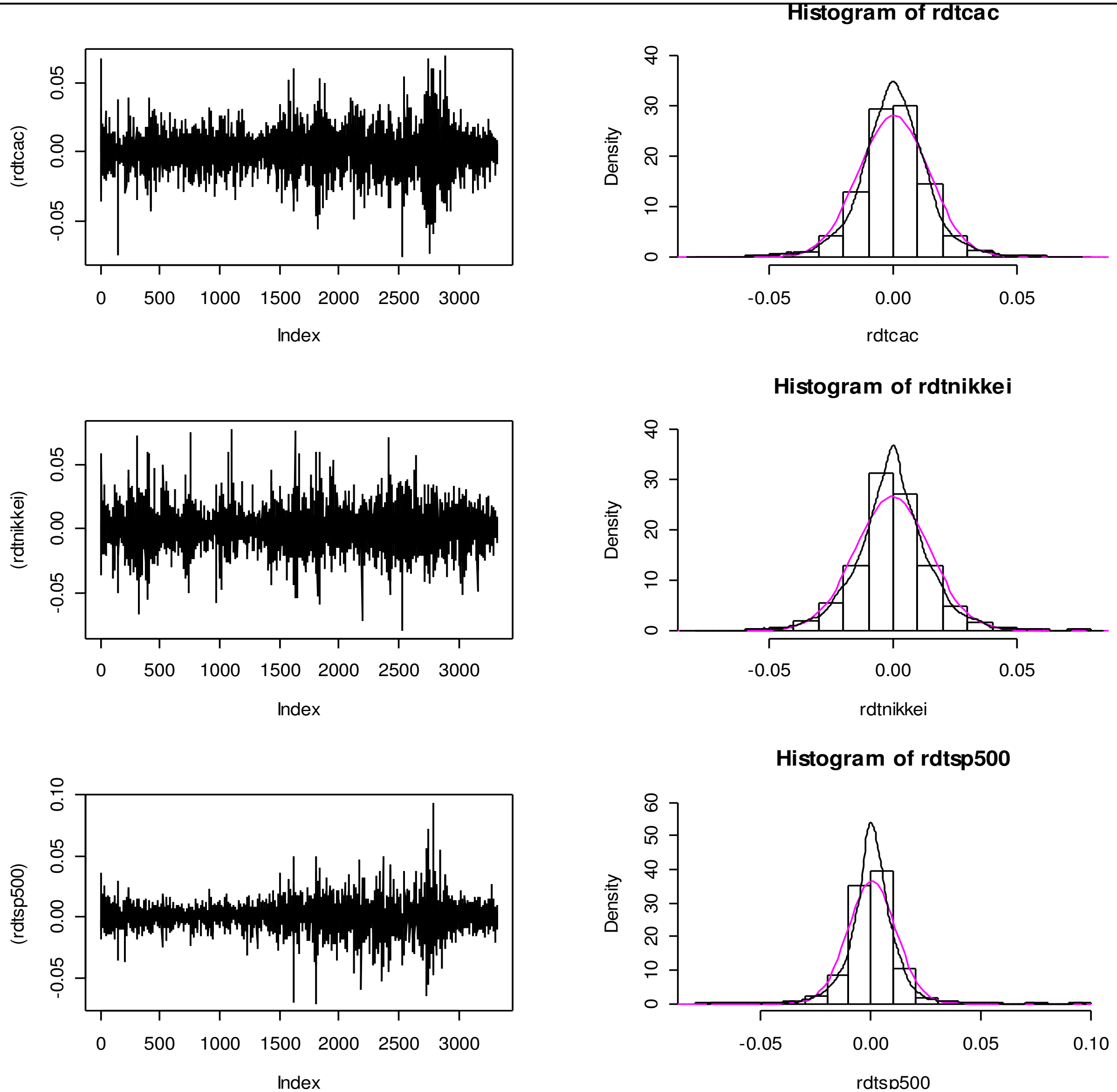
Plot of Simulated Returns
0.06
0.04
0.02
0.00
-0.02
-0.04
-0.06
0
2000
4000
6000
8000
10000
Histogram of simu
1% of the distribution
0
5
15
20
25
-0.06
-0.04
-0.02
0.00
0.02
0.04
0.06

Each has pros and cons

Method	Strengths	Weaknesses
Historic	Very easy to implement	Assumption that the distribution is stable over time
Parametric	Simple	Most of the time assumes normality Dependence parameters (var-cov) are predictable Non linearities generate difficulties (even in the Cornish Fisher approach, results are not robust)
Simulation	Non linearities are taken into account	Time consuming. Portfolios are priced for each path

- The historical approach (HA) is extremely sensitive to the choice of the sample. From this approach it is impossible to obtain losses that are not observed in the sample. The probability distribution of the variation of the risk factors is supposed to be constant over time and remains valid in the future
- Nevertheless, HA is the only distribution free method. For the other two, the choice of the distribution has a preeminent influence on the calculation of the risk measure.

- So for the Analytic and Simulation methods, the choice of distribution has a preeminent influence on the calculation of the VaR
- There is no consensus on what distribution to use with financial returns.
- Calculation of the third and fourth order moments performed on financial series often show that the latter are far from normal.


Histogram of rdtcac
(rdtcac)
0.05
0.00
-0.05
0
500
1000
1500
2000
2500
3000
Index
Density
rdtcac
Histogram of rdtnikkei
(rdtnikkei)
Index
Density
rdtnikkei
Histogram of rdtsp500
(rdtsp500)
0.10
Index
Density
rdtsp500

- Several limitations
  - Several methods to obtain this measure. Each of these leads to significant discrepancies
  - Intrinsic issues (not sub additive measure, what about losses beyond the VaR ?, etc.)

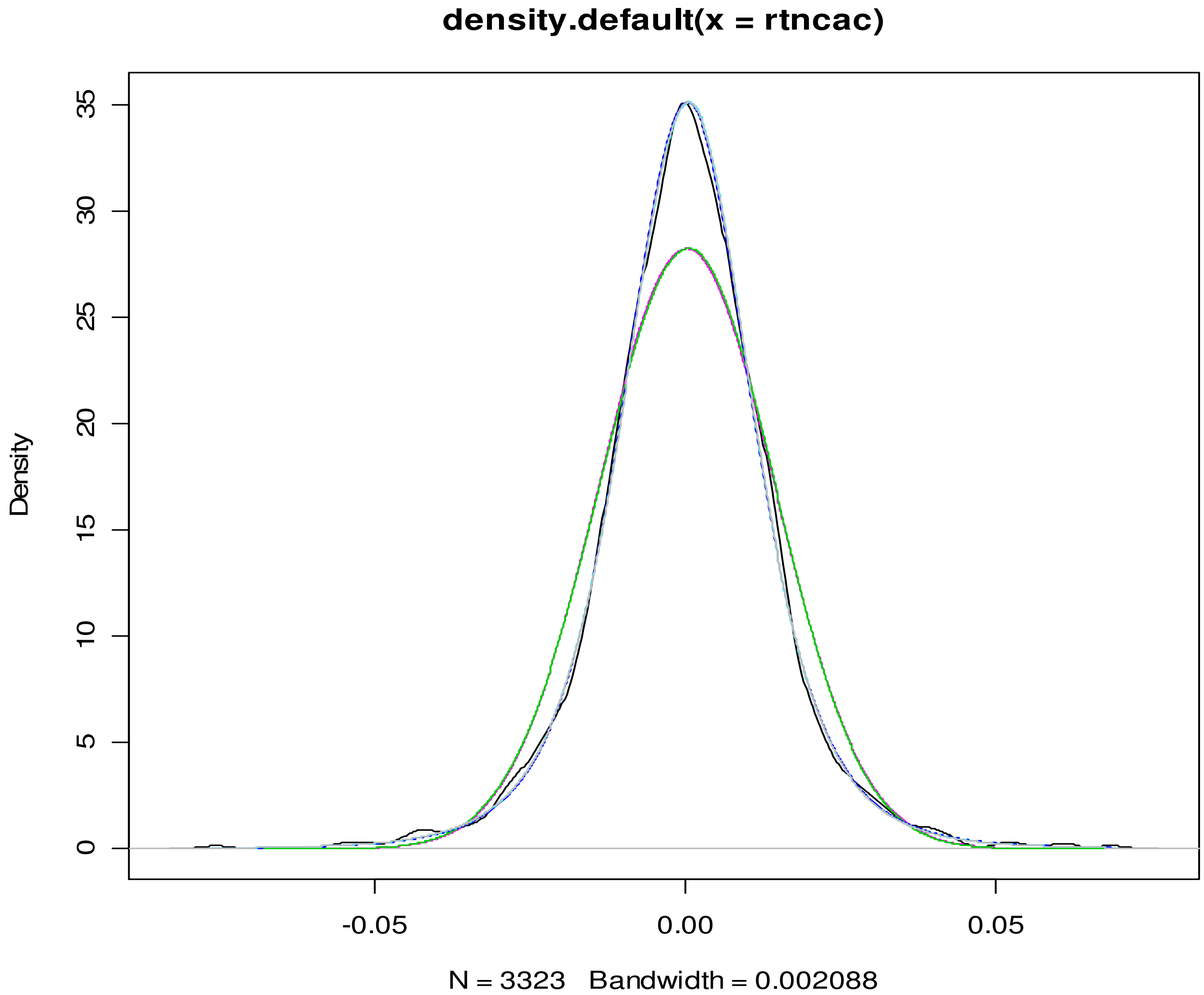
- To reduce those limits we can use measures beyond VaR.
- Expected Shortfall corresponds to the average loss in the worst 100%α cases, i.e the average loss when losses are greater than the VaR
- Even if we consider B-VaR measures the choice of the best approach remains, so do the discrepancies

Toward integration of Extreme Values

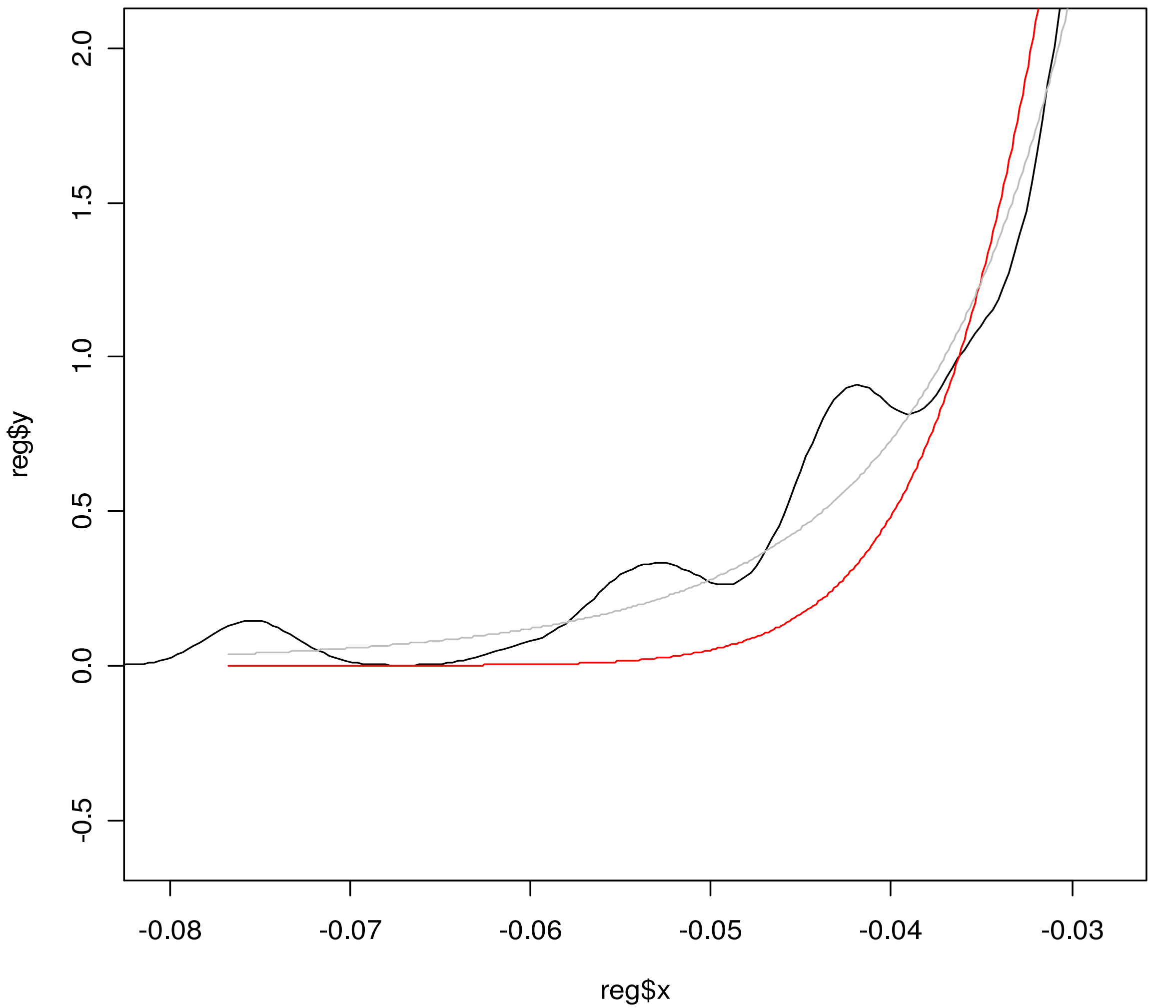
- Risk departments of banking institutions work frequently on “atypical scenarios” = catastrophic scenarios which are exceptional but plausible
- It does not integrate the probability of the extreme scenarios occurring
- It cannot be used to obtain VaR
- So there’s a real need to integrate extreme events better.

- Calculation of the traditional VaR works very well if we stay near the centre of the distribution, it remains much more delicate to use for high risk threshold.
- The use of fat tailed distribution (Pareto's, Student's, skew Student's, etc.) improves the accuracy of the VaR but does not solve it completely

The use of fat-tailed distribution improves the accuracy of the fit…



… particularly in the centre of the distribution but not in the tail

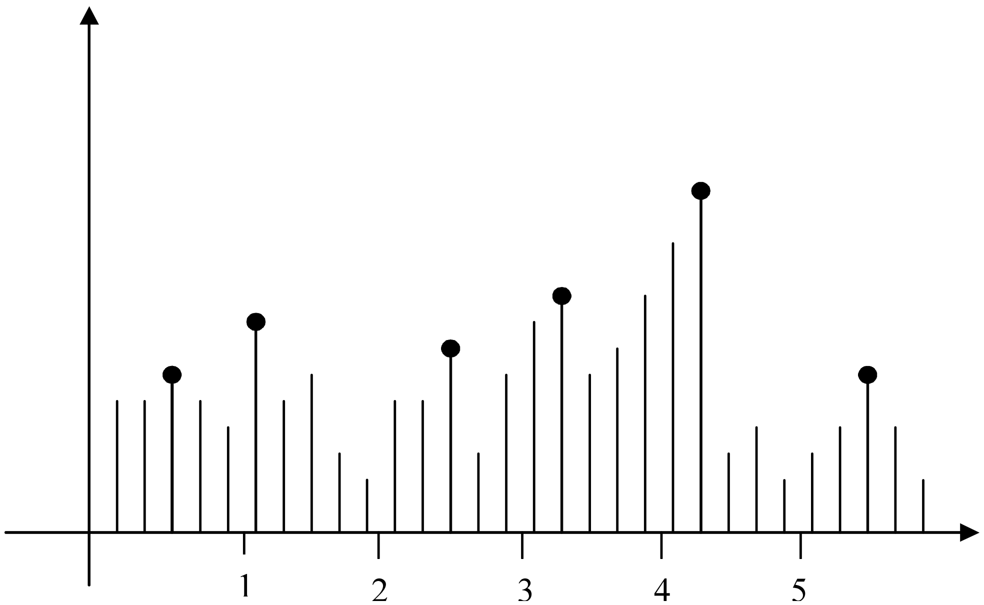


- As it is impossible to completely fit observed data with parametric distributions, why not focus on extreme values only?

The Extreme Value Approach(es)

Block Maxima (1)

- The variable under consideration is the largest loss observed on a set of samples of identical size



Selection of maxima on a window of 5 consecutive trading days

Block Maxima (2)

- X1, …, Xn is a series of iid random variables
- They all have the same cumulative distribution, i.e F(x)=P(Xi≤x)

Series of the largest loss:

$$M_n = Max\{X_1, X_2, ..., X_n\}$$

Cumulative distribution of Mn:

$$P(M_n \leq x) = P(X_1 \leq x, ..., X_n \leq x)$$
$$= \{F(x)\}^n$$

Block Maxima (3)

- To reduce the impact of a specific choice of F(x), we look into the asymptotic behavior of Mn

$$\lim_{n\to\infty} P\left(\frac{M_n - \mu_n}{\sigma_n} \leq x\right) = G(x)$$

Block Maxima (4)

- If G(x) is not degenerated it can only correspond to one of these three distributions (Fisher Tippett Theorem):

$$\text{Gumbel} \quad : \Lambda_{\alpha}(x) = \exp\left[-\exp\left(\frac{-x+\mu}{\sigma}\right)\right], \quad x \in \Re$$

$$\text{Fréchet} \quad : \Phi_{\alpha}(x) = \begin{cases} 0, & x \leq 0 \\ \exp\left[-\left(\frac{x-\mu}{\sigma}\right)^{-\alpha}\right], & x > 0 \end{cases}$$

$$\text{Weibull} \quad : \Psi_{\alpha}(x) = \begin{cases} \exp\left[-\left(-\left(\frac{x-\mu}{\sigma}\right)\right)^{\alpha}\right], & x \leq 0 \\ 1, & x > 0 \end{cases}$$

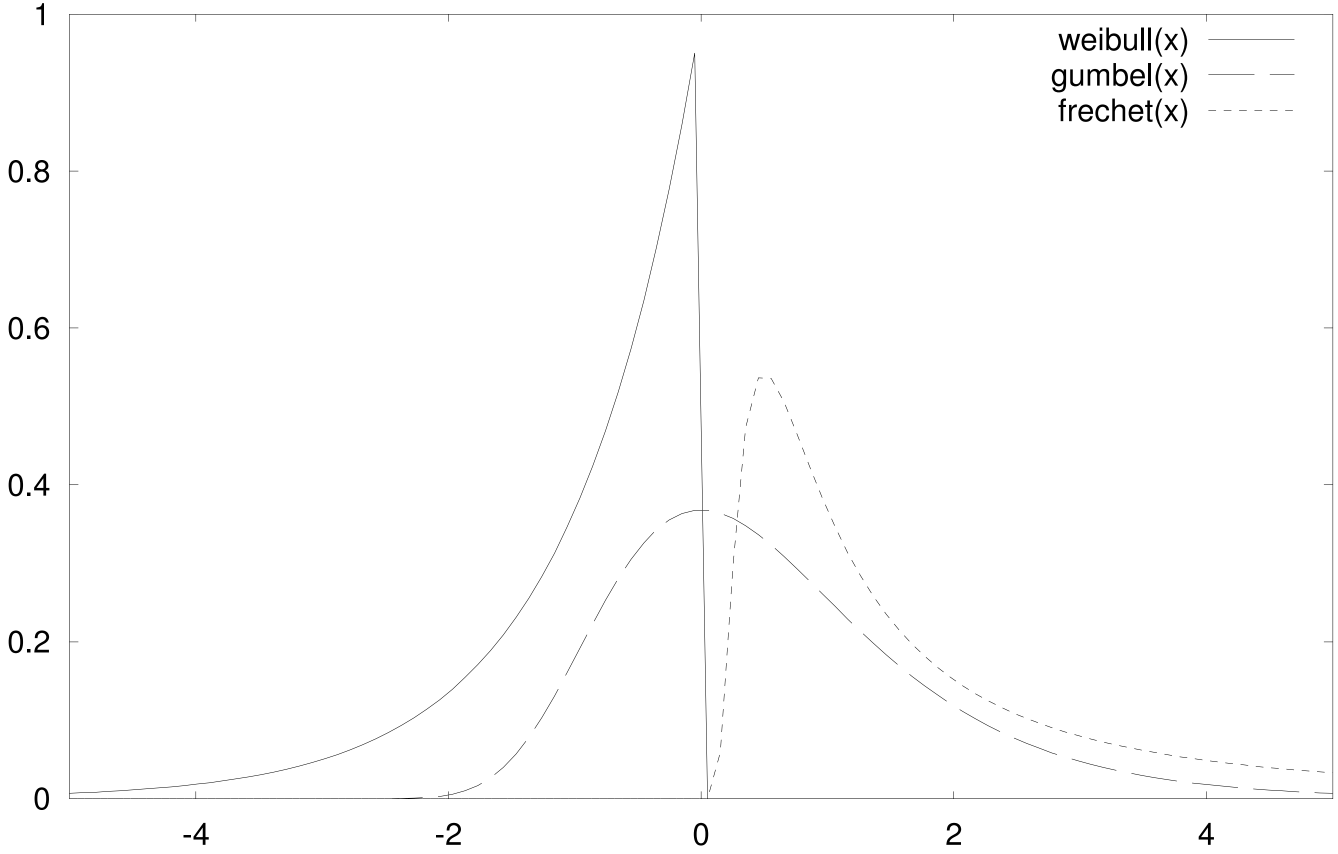
Block Maxima (5)

- Those distributions could be synthesised by the following distribution (Jenkinson – Von Mises), simplifying parameter estimation

$$G(x)=\begin{cases}\exp\left[-\left(1+\xi\left(\frac{x-\mu}{\sigma}\right)^{-\frac{1}{\xi}}\right)\right], & 1+\xi\frac{x-\mu}{\sigma}>0,\ \xi\neq 0\\ \exp\left[-\left(1+\xi\left(\frac{x-\mu}{\sigma}\right)\right)\right], & x\in\Re,\ \xi=0\end{cases}$$

$$g(x)=\frac{1}{\sigma}\left[1+\xi\left(\frac{x-\mu}{\sigma}\right)\right]^{-\frac{1+\xi}{\xi}}\exp\left[-\left(1+\xi\left(\frac{x-\mu}{\sigma}\right)\right)^{-\frac{1}{\xi}}\right]$$

Block Maxima (6)



Distribution function of the Extreme distribution ξ=1, μ=0, and σ=1

Block Maxima (7)

- The Weibull distribution, unlike the Fréchet and Gumbel distributions, is not used in finance as it is characterized by thin tails
- VaR is very simple to obtain from the extreme value distribution

$$VaR = G^{-1}(q)$$
$$= \mu - \frac{\xi}{\sigma}\left[1 - \left(-\ln(q)\right)^{-\xi}\right]$$

Block Maxima (8)

- Several methods exist to estimates the parameters of the GEV.
- The first one consists of a traditional maximum likelihood estimation
- The second is constructed around estimators of the extremal index (Pickands and Hill)

Block Maxima (9)

In the MLE:

- There is no explicit solution for $\hat{\theta}$
- Properties of MLE do not hold for $\xi \leq -1/2$
- ML estimations suppose large sample size

With direct estimation:

The Pickands or Hill estimators are sensitive to the choice of one exogenous parameter (k) . If k is too large, data are no longer in the tail, whereas if k is too small, the estimator will vary.

Block Maxima (Conclusion)

- Mainly used for large number of observations
- With 3 years of historical data on a single risk factor and windows of 25 trading days we only obtain 30 extrema
- Selection of the mode in this approach could lead to significant information loss

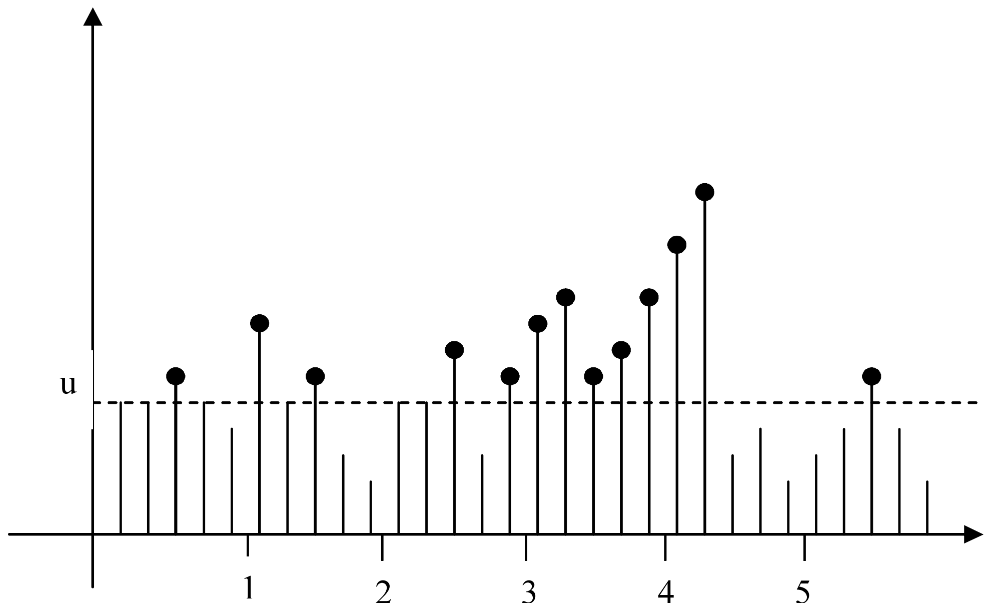
Block Maxima (Conclusion 2)

- Apart from the maxima all the other values are discarded from the analysis, i.e. we do not integrate the dispersion of the returns (whether all the returns are close to the maximum or not is not taken into account)
- So this approach is theoretically interesting but difficult to carry out in practice

Peak over Threshold (1)

- We focus our analysis on losses above a high threshold
- Two approaches could be distinguished:
  - A semi-parametric approach constructed around the Hill estimator
  - A parametric approach based on the Generalized Pareto Distribution (easier to implement)

Peak over Threshold (2)



Peak over Threshold (3)

$$F_u(y) = P\{X - u \le y / X > u\}$$
$$= \frac{P(u < X \le y + u)}{P(X > u)}$$
$$= \frac{F(u+y) - F(u)}{1 - F(u)}$$

- When u is high, Fu converges towards a GPD

$$G_{\xi,\beta}(x) = \begin{cases} 1 - (1 + \xi x / \beta)^{-1/\xi} & \text{if } \xi \ne 0 \\ 1 - \exp(-x / \beta) & \text{if } \xi = 0 \end{cases}$$

With $\beta>0$, $x>0$ when $\xi\geq 0$ and $-\beta/\xi \geq x \geq 0$ when $\xi<0$

Peak over Threshold (4)

- Within our framework only the case where $\xi \geq 0$ should be selected (fat tail distributions)
- VaR formula is then given by:

$$VaR_q = u + \frac{\beta}{\xi}\left(\left(\frac{n}{N_u}(q)\right)^{-\xi} - 1\right)$$

With Nu the number of data greater than the threshold and n the total number of data. So n/Nu is an historical estimator of F(u)

Peak over Threshold (5)

- Even if this approach is both theoretically accurate and operationally useful, it can only be validly used for single risk factor.
- Then we must consider the P&L of the portfolio rather than individual sources of uncertainty
- One alternative is to use copula

Conclusion

Our conclusion is threefold

- You should complete traditional Risk assessment with ES
- Even if you use fat-tailed distributions you can “miss” something
- Explicitly modelling Extremes will result in gains

Selected references

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ANNEXES

Estimators for ξ

- Pickands

$$\hat{\xi}^{P}_{k,n} = \frac{1}{\log 2} \log\left(\frac{X^*_{n-k:n} - X^*_{n-2k:n}}{X^*_{n-2k:n} - X^*_{n-4k:n}} \right)$$

- Hill

$$\hat{\xi}^{H}_{k,n} = \frac{1}{k} \sum_{i=1}^{k} \log X^*{}_{i-1:n} - \log X^*{}_{k:n}$$

Where X^* are the ordered data

Maximum Likelihood Estimation

1. ξ ≠0

$$l(\theta) = \log L(\mu, \sigma, \xi)$$

$$= -n \log \sigma - \left(\frac{1}{\xi} + 1\right) \sum_{i=1}^{n} \log\left(1 + \xi \frac{M_n - \mu}{\sigma}\right) - \sum_{i=1}^{m} \left(1 + \xi \frac{M_n - \mu}{\sigma}\right)^{-\frac{1}{\xi}}$$

2. ξ =0

$$l(\theta) = -n \log \sigma - \sum_{i=1}^{n} \exp\left(-\frac{M_n - \mu}{\sigma}\right) - \sum_{i=1}^{m} \left(\frac{M_n - \mu}{\sigma}\right)$$

$$(\hat{\theta}) = \arg\max_{\theta} l(\theta)$$