EDHEC RISK AND ASSET MANAGEMENT RESEARCH CENTRE
Edhec -1090 route des crêtes - 06560 Valbonne - Tel. +33 (0)4 92 96 89 50 - Fax. +33 (0)4 92 96 93 22
Email: research@edhec-risk.com – Web: www.edhec-risk.com

History of the Risk Concept and Risk Modeling

October, 2004

Jean-Christophe Meyfredi
EDHEC Business School

History of the Risk Concept and Risk Modeling

Jean-Christophe MEYFREDI
EDHEC Business School
58, Rue du Port
59046 Lille Cedex

October 18, 2004

1 Introduction

What is risk? The answer is far from simple. The definition depends on the context and is highly subjective. A first attempt is to define risk as the possibility of something unexpected occurring. But what could the constituents of those expected and unexpected events be? Again there is no single answer. The field of finance is a symptomatic example where risk is multiform. It is usual to distinguish between market risk, credit risk, liquidity, operational and legal risks. All those risks could generate losses that would be more or less prejudicial for an institution or for a single investor.

Moreover, even if this definition is very close to the one of uncertainty we must distinguish between those two concepts (Knight (1921)). Under Knight's definition, uncertainty corresponds to a situation where the decisions of every economic agent depend on exogenous factors whose states could not be predicted with certainty. Only when uncertainty could be quantified, i.e. when it is possible to assign a probability distribution, can we speak about risk. Finally dealing with risk requires answers to two questions: How much can I lose and what is the probability that this loss will occur?

Risk constitutes an important field of research that has been of increasing interest in the last ten years. There are at least two reasons for handling risk. Firstly, there is a necessity for the decision-maker to act with full knowledge of the facts. Secondly, risk must be limited and also managed. A bad assessment could leads to bankruptcy or even to a systemic crisis. Current events are full of outstanding examples: the stock market crash of 87, Barings, Orange County and LTCM are some typical cases.

For all these reasons, risk measurement plays a central part in risk management, which is a quite recent speciality but now indispensable. Risk measurement is traditionally examined either indirectly by studying the risk preferences of decision-makers, or directly by defining a risk measure that is independent of the risk preferences that could be useful for selecting the right opportunity. We have chosen to focus only on this second approach for several reasons. The first reason is the lack of information we have on the utility functions of each decision maker. Moreover, the utility function is closely related to the initial wealth of the decision-maker and is not strictly concave or convex in reality. So the modeling that follows from this approach is still essentially theoretical. The second reason is that we can easily go from one approach to the second (see for example Jia and Dyer (1996)), but the second is easier. The third reason is that when risk management wishes to forecast the evolution of the different risk factors that could have an impact, a measure is required to value the risk the institution has to face. So determining the 'right' risk measure appears essential for making an efficient decision in many respects.

Faced with the numerous risk measures that have been created in the financial area we have chosen to restrict our study to general risk measures, that is to say risk measures that could be used whatever the assets considered. Several specific risk measures exist and can be useful[1] but they are not within the scope of this study. We limit our analysis to risk

[1] Beta for individual shares or portfolios of stocks, gaps for treasury management, duration or convexity for bonds, greeks for options, etc.

measures of individual assets or portfolios taken as a whole, so measures of dependence like covariance, copula and so on, are not the subject of this analysis.

Let's see how risk measurement has evolved over the past 50 years and where it is heading.

2 Traditional risk measures

Most of the time, traditional literature lists the various risk measures without any attempt to organize a classification. Nevertheless several taxonomies can be used to distinguish between them. We choose to distinguish between measures of dispersion and downside risk measures (sometimes classified as safety risk measures, see for example GIACOMETTI AND ORTOBELLI-LOZZA (2004)). This decomposition presents the advantage of integrating almost all of the existing risk measures.

In finance, risk is most frequently calculated from a random variable X representing the future net worth of a position or the relative or absolute changes in value of an investment. We will use the same representation here.

2.1 Measures of dispersion

Probability-weighted dispersion constitutes the first way of dealing with risk measurement. In this case, those measures of dispersion could be divided into three classes. The first class groups together measures of the distance between some representative values. The second one covers all the measures obtained from deviation of each data from a reference point (also called symmetric measure of risk). The last class is made up of any measures obtained from the deviation of all the data among itself.

2.1.1 Distance between representative values

Three main measures constitute this group:

- The range, which is simply the difference between the highest and lowest value taken by the variable under consideration
- The interquartile range, which gives the difference between the lowest and highest quartiles and therefore contains one-half of the total population[2]
- The maximum loss (or maximum drawdown), corresponding to the worst loss obtained from a sample

Unfortunately, those measures give no information about the dispersion inside the range and as a consequence, have very limited utility in the case of risk management. We only get an intuitive idea of the distribution spread.

2.1.2 Deviation from a central value

By construction, the sum of the deviation from the mean is zero. To avoid the negative and positive deviation offsetting each other, two main approaches can be used. The first is to sum the squared deviations. The second consists of calculating the sum of the absolute deviations from the mean. From the seminal work of MARKOWITZ (1952), several risk measures have been defined theoretically and used in practice. Without any contest, the variance and its square root, i.e. the standard deviation, constitute the most widely employed measures. The variance is defined as the expected value of the squared deviations of the data values (returns) from the mean, and thus simply measures the dispersion of the estimates around their mean value.

$$
\begin{aligned}
Var(X) &= E\left[(X - E(X))^2\right] \\
&= \begin{cases} \sum\limits_{R} (X - E(X))^2 f(x) \text{ if } X \text{ is discrete,} \\ \int_{-\infty}^{+\infty} [X-\mu]^2 f(x)\, dx \text{ if } X \text{ is continuous} \end{cases},
\end{aligned}
$$

where $f(.)$ corresponds to the probability distribution function and $\mu = \int_{-\infty}^{+\infty} x f(x) dx$ is the mean. In practice, standard error is more useful because it is expressed in the same unit as the data under consideration. Most of the time, in a financial context, this measure is also called volatility. For the rest of the paper we will present only general and continuous versions of the different risk measures.

The Expected Absolute Deviation (sometimes called the Mean Absolute Deviation) is the sum

[2] Other risk measures derive from this one. Thus the Semi-Interquartile Deviation corresponds to one half of the interquartile range and the coefficient of quartile variation is the interquartile range divided by the second quartile.

of the absolute values of the deviations from the mean (of course this measure could be adapted to any other threshold, like 0, the distribution median or its mode for example)

$$EAD = \int_{-\infty}^{+\infty} |X - \mu| \, f(x) \, dx.$$

Other deviation measures for a central value could be constructed[3]. However, this class of risk measure exhibits the major drawback of implicitly assuming distribution functions characterised by specific particularities such as symmetry, and/or adapted to a very restrictive utility function like the quadratic one. They also do not take account of a loss appearing with small probabilities.

2.1.3 Deviation among data

The Mean Difference is given by the average of the differences of all possible pairs of variate-values, taken regardless of their sign.

$$\Delta = \frac{1}{2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |x - y| \, f(x) f(y) dx dy,$$

where y is an *iid* copy of x. In a discrete framework, we get:

$$\Delta = \frac{1}{2N(N-1)} \sum_{j=0}^{N} \sum_{\substack{k=0 \\ j \neq k}}^{N} |x_j - x_k| \, f(x_i) f(x_k).$$

With the continuous version of the measure, the integral could converge when the variance doesn't converge. This measure is also used in Gini's coefficient of concentration, which is the ratio of the mean difference and twice the mean[4].

Yitzhaki (2002) shows that this measure shares many properties with the variance but is more informative for distributions that depart from normality.

[3] For example, see Ortobelli, Huber, and Schwartz (2002) who propose a new dispersion measure that could be used if the returns follow a joint α-stable sub-Gaussian distribution.

[4] The principle of this ratio is to normalise the dispersion parameter by a parameter of location, as is done for example with Pearson's coefficient of variation. In this case the ratio is independent of the variate scale and can be used to compare dispersions in different populations.

The variance has also been adapted to measure deviation not from the mean but from the return of a specified benchmark. This leads to the tracking error measure.

With those deviation measures, data above and beyond the threshold are equivalently considered. But obviously, investors are more affected by losses than by gains! To penalise losses more than gains, an alternative focusing only on downside risk has been developed. Then it becomes possible to consider skewed distributions.

2.2 Downside risk measures

Most of the literature dedicated to decision making presents the notion of risk as the failure to reach a specific target return. In fact, the main shortcoming of the previous approach, even if it is very appealing, is that deviations were considered in the same way both above and below the reference point (like the return average for example). Nevertheless, investors are most concerned by type of risk that correspond to negative returns rather than positive ones. Therefore, we now focus on the various asymmetric measures of risk and more specifically the downside risk measures.

2.2.1 Below a reference point risk measures

The Safety First measure, developed by Roy (1952) offers a first response to the main drawback of deviation measures. It consists of fixing the probability of obtaining a loss below a specified intolerable target level (”disaster level”):

$$PL : P(X < \tau) = \int_{-\infty}^{\tau} f(x) \, dx.$$

Expected value of loss Domar and Musgrave (1944)

$$EVL = -\int_{-\infty}^{0} x f(x) \, dx$$

The semi-variance (Markowitz (1959)) corresponds to the expected value of the squared negative deviations from the mean. With the same notation as before,

$$SV(X) = E\left[(\min\{X - E(X), 0\})^2\right] = \int_{-\infty}^{\mu} (\mu - X)^2 f(x)\, dx \quad (1)$$

We can note that with a symmetrical distribution, the semi-variance is proportional to the variance. As it is most of the time assumed that returns follow a normal distribution, there is no advantage to using the semivariance and thus we come back naturally to the use of the variance. Unfortunately most empirical studies showed that financial returns are far from normal, especially for long time horizons. For asymmetrical returns, like options for example, we gain by using measures that can consider skewed and fat tailed distributions. This could be done by measuring the third (skewness) and fourth (kurtosis) moments of the distribution under consideration.

Two concepts are very close to this definition. The first one is the concept of truncated variance. This measure is defined as the standard deviation of negative values only and not simply those below the mean. Of course, the two measures give the same results for distributions with a skewness equal to zero. We represent its continuous version[5] :

$$TV = \int_{-\infty}^{0} \left(X - \mu^-\right)^2 f(x) dx \quad (2)$$

with $\mu^- = \int_{-\infty}^{0} x f(x) dx$.

The second measure close to the semi-variance is the Lower Confidence Limit, proposed by Baumol (1963), also called the standard-deviation corrected mean (see for example Gaivoronski and Pflug (2000)). In Baumol's paper, he criticised a decision based only on standard deviation without linking it to the expected return. That is why he develops an "improvement" to the Markowitz approach based on the standard deviation. This measure is given by the expected return minus k times the standard deviation. One advantage of this measure is that it could be linked to the investor's risk preferences[6].

Almost all of the previous risk measures could be generalised by the general class of three-parameter risk measures (L) developed by Stone (1973):

$$L = \int_{-\infty}^{q} |X - \tau|^{\gamma} f(x)\, dx,$$

where q corresponds to the range parameter that indicates which deviations are to be included in the risk measure, τ is the value level from which we want to measure the deviations, and γ ($\gamma \geq 0$) measures the relative impact of large and small deviations[7].

We can see that this last formula encompasses various risk measures. For example, if we set the γ parameter to 2, $q = +\infty$ and τ to the expected return μ, we come back to the variance measure. With the same parameters for γ and τ but with $q = \mu$, we obtain the semivariance. With $q = +\infty$, $\gamma = 2$ and τ corresponding to the return (variable) of a benchmark, we get the tracking error, and so on.

Various thresholds from which deviations are calculated could be used. Naturally, we can choose the mean, but also zero, the initial wealth, the distribution median, the distribution mode, or whatever threshold that could be useful for the decision maker, such as, for example the risk-free rate or any other benchmark (not only deterministic but also stochastic).

To include risk measures like the standard deviation, Stone (1973) proposes a second class of three-parameter risk measures, which corresponds to the k^{th} root of the last measure.

The main drawback to the general class of risk measures proposed by Stone is that risk is "not necessarily a monotone increasing function" of γ. So, with this measure, we can face situations where the risk aversion is higher for lower values of γ. Bawa (1975) and Fishburn (1977) propose an adaptive measure, the lower partial moments, by reducing the number of parameters to two:

$$LPM(\tau) = E\left[(\max\{\tau - X; 0\})^{\gamma}\right] = \int_{-\infty}^{\tau} (\tau - X)^{\gamma} f(x) dx$$

[5] The general formulation is : $TV = E\left[(\min\{(\min(X;0) - E(\min(X;0)));0\})^2\right]$.

[6] Even in case of non gaussian distributions the link with risk preference could be maintained with Chebychev's inequality.

[7] Even though all positive values could be used for γ, γ is set to $0, 1$ or 2 in general.

With this risk measure, the range parameter is set equal to the threshold from which the deviations are calculated. It now becomes possible to link this risk measure to the investor's aversion to risk, completing the work done by JEAN (1975) on Stone's measure. Here, risk averse investors will choose a γ coefficient greater than 1 (giving more importance to large deviations), those neutral to risk will chose a γ equal to 1 (all deviations will be weighted equally) and γ will be lower than one for investors who dislike risk.

We have seen that variance has played and still plays an important role in risk modelling. To capture the excess kurtosis that financial series generally exhibit, FAMA (1965) investigates the Markowitz framework in the case of the distribution of returns having an infinite variance, and more particularly in the case of the stable paretian distributions[8] with a characteristic exponent lower than two. Unfortunately, it is difficult to give "operational or computational meaning of the risk factor in all cases"[9]. Even if this class of distribution exhibits fat tails, it also leads to various drawbacks like for example the lack of finite second moments in most cases, but empirical analyses tend to show that variance estimates converge when sample size increases. This could explain why stable paretian distributions (other than normal) seem to have disappeared in financial studies.

For many years, decisions under uncertainty were taken focusing on the variance criterion. Progress in numerical algorithms and computer technology led to the improvement of optimisation programs and the development of other measures that progressively became the new standards for risk management.

2.2.2 Value at Risk and beyond

Through its role in preventing the reoccurrence of a financial disaster like that encountered by Barings, for example, and particularly those that have arisen since the beginning of the 90s, Value-at-Risk (VaR) has rapidly become the most useful risk measure. It has developed swiftly since its definition first appeared in a report written by the Group of Thirty[10] in 1993[11]. The most widely-used definition of VaR corresponds to the possible level of loss that can be sustained in normal market conditions for a given period[12] and for a given confidence level[13]:

$$\begin{aligned} VaR_\alpha &= -\sup\{x|P(X<x)<1-\alpha\}, \\ VaR^\alpha &= -\inf\{x|P(X<x)<1-\alpha\}. \end{aligned}$$

So VaR is the $(1-\alpha)\,\%$ quantile of the distribution. As $VaR^a = VaR_\alpha$ for a continuous distribution, we can in this case write:

$$VaR_\alpha = -F_X^{-1}(1-\alpha),$$

where F_X^{-1} denotes the inverse of the cumulative distribution function.

As such, if a portfolio exhibits a VaR of USD 250,000 for ten days at 99%, this simply means that the portfolio can lose more than USD 250,000 for only one percent of cases for those ten days. By extension, we can formulate this result as follows: the portfolio can lose more than USD 250,000 only 2.5 times a year.

The fast development of the VaR can be linked to the simplicity of a concept that was initially created for risk managers in credit institutions, but is now used to communicate with clients, shareholders and regulators. Unfortunately the VaR concept exhibits some undesirable drawbacks.

A first difficulty arises with VaR estimation. Three methods could be used to obtain VaR (historical, parametric and simulations)[14]. Each of those has strengths and weaknesses. However, these difficulties are only found in the methodology for obtaining the VaR (choice of the method or

[8] We recall that a non degenerate random variable X is stable if for positive numbers a and b there exists a positive number c and some number d such that :
Law$(aX_1 + bX_2)$ =Law$(cX + d)$,
with X_1 and X_2 independent random copies of X.

[9] p 416.

[10] Group of Thirty, "Derivatives: Practices and Principles", New York, July 1993.

[11] Even though the term of VaR was only defined in 1993, the concept has been used in practice for several years, notably by the SEC since 1980.

[12] The holding period is traditionally between one day and two weeks.

[13] α is in general set to at least 95%. VaR is sometimes given in terms of level of significance.

[14] See for example JORION (2000), PENZA AND BENSAL (2001), HOLTON (2003) or MEYFREDI (2003).

data used). Unfortunately, whatever the methodology employed, the VaR is subject to two more limitations. Firstly the VaR doesn't take the presence of fat tails in financial series into account. Secondly the VaR could go against the diversification principle.

How does one choose between the various risk measures presented above? We saw that some of them could be easily dismissed. For others the decision is not so simple. There is thus a real need for a selection criterion that could be used to select the "right" measure. ARTZNER ET AL. (1997) and (1999) were the first to develop an axiomatic approach to risk measurement in the area of finance. They aim to define the minimal set of properties that a risk measure must satisfy. Defining X as the final net worth of a portfolio (random variable) and $\rho(X)$ as the risk measure, the risk measure must fulfill the following properties:

- Translation invariance: $\alpha \in \mathbb{R}$, $\rho(X+\alpha.r) = \rho(X) - \alpha$.
- Subadditivity: $\rho(X+Y) \leq \rho(X) + \rho(Y)$
- Positive homogeneity: if $\lambda \geq 0$, $\rho(\lambda.X) = \lambda.\rho(X)$
- Monotonicity: if $X \leq Y$, then $\rho(Y) \leq \rho(X)$

Whenever a risk measure satisfies these four axioms it is defined as a "coherent measure of risk".

Translation invariance means that adding cash to a portfolio decreases its risk by the same amount (in Artzner et al.'s paper, risk is considered as the final net worth). This first axiom also leads to a very interesting conclusion: if we add $\rho(X)$ to the initial position, the resulting position is risk neutral. This axiom leads to the rejection of deviation measures as coherent ones[15]. The second axiom, subadditivity, corresponds to the fact that "a merger doesn't create extra risk"[16]. Thus, it reflects the famous diversification principle. Another way to explain this property is that one cannot reduce the risk of a portfolio for example by splitting it into numerous sub-portfolios. This property is very appealing for regulatory purposes. In fact, a subadditive measure is a conservative one as the sum of the risk of various subportfolios is always lower or equal to the total risk taken. By imposing such a property, regulators can be protected from possible attempts at manipulation. The third axiom reflects the fact that when we double the size of all positions in a portfolio, the risk of the portfolio will be twice as large. Finally, the monotonicity axiom means that if project Y is preferred to project X, then the risk of X is higher than that of Y.

Artzner et al.'s paper allows the 'right' risk measure to be selected and constitutes a major advance in risk measurement theory and practice. However the most widely used measure, the VaR, fails to respect the four axioms[17]. Artzner et al. propose a coherent risk measure, the Expected Shortfall(ES)[18], corresponding to the expected loss in the worst α-cases.

$$ES_\alpha = -\frac{1}{1-\alpha}\int_0^{1-\alpha} F_X^{-1}(p)\,dp.$$

This axiomatic approach has generated several improvements. For example, the lack of liquidity could leads to a violation of the subadditivity and positive homogeneity axioms. Those axioms could then be replaced with a less restrictive one allowing the risk of a position and its size to be linked in a non-linear way (see HEATH (2000), CARR, GEMAN, AND MADAN (2001), or FRITTELLI AND ROSAZZA-GIANIN (2002)):

- Convexity: $\alpha \in [0,1]$, $\rho(\alpha X + (1-\alpha)Y) \leq \alpha\rho(X) + (1-\alpha)\rho(Y)$.

Non-convex risk measures[19] can exhibit several optima and cannot be used for optimisation. This is not the case for coherent risk measures which constitute a subset of convex risk measures.

DELBAEN (1998) extends this seminal work by ARTZNER *et al* in a finite probability space, to more general probability spaces.

[15] All the deviation measures are shift invariant, i.e. $\rho(X+\alpha.r) = \rho(X)$.

[16] Artzner, Delbaen, Eber, and Heath (1999), 9^{th} page.

[17] VaR only respects the different axioms for elliptic distributions.

[18] Expected Shortfall is also called Conditional Value-at-Risk and Expected Loss. For discrete distributions, Expected Shortfall must be distinguished from Tail Conditional Expectation and Worst Conditional Expectation (see ACERBI AND TASCHE (2002)) which correspond to the expected loss beyond the VaR.

[19] Those which failed to respect the previous axiom, i.e $\frac{\partial^2\rho}{\partial\alpha^2} \leq 0$.

Another improvement is the generalisation of the Expected Shortfall by Spectral Risk Measures[20]. This new class of risk measures is constructed as to validate the axioms of coherency. This class groups together measures of the type:

$$M_{\phi,c}(X) = cES_0(X) - (1-c)\int_0^1 \phi(p) F_X^{-1}(p)\, dp,$$

where $ES_0 = \inf\{x|F_X(x) > 0\}, c \in [0,1]$ and ϕ a monotonous positive function whose integral between 0 and 1 is equal to one. A complete description of those risk measures can be found in Acerbi (2004).

The different axioms used to characterise a risk measure are closely linked to the applications of the risk measure (see Goovaerts and Vyncke (2004) or Goovaerts, Kaas, and Dhaene (2002)). In fact, the four previous axioms are useful in finance, but some of them could be of limited interest if used in another specific area such as insurance. This is particularly true for the subadditivity axiom, as the risk taken to insure two flats in the same building could be higher than the sum of the risk two companies will take for those two flats. This example leads to the conclusion that the choice of a risk measure is a specific one depending on the use we expect for it. Several other axiomatic approaches have been developed in the context of insurance (see for example Wang, Young, and Panjer (1997)) and have led to the definition of specific risk measures[21].

Other risk measures have been developed in specific context as for example in communication, in psychology[22], etc. However these risk measures have limited interest in decision-making because they are not linked with preference models.

3 Conclusion

In this document we concentrated on risk measurement by analyzing several well-known risk measures. We have chosen not to develop specific risk measures, or the various measures of dependence, in order to focus on more general measures that could be used whatever the risk factor considered.

We have seen that several risk measures exist, but they do not necessarily offer an answer to the needs of risk management. The choice of measure depends on the intended use. Thus, the choice of a risk measure stems from the axioms a measure has to respect. One risk measure seems to have generated, at least in finance, a consensus as to its usefulness: the Expected Shortfall. With this measure, it is now possible to handle extreme events and to measure risk in a more precise and easier way. Moreover, new developments in finance and insurance revolve round this issue.

[20] A spectral risk measure is of the form $M_\phi = -\int_0^1 \phi(x) F^{-1}(x)\, dx$, where ϕ is a weight function reflecting the degree of risk-aversion.

[21] For example the class of distorsion risk measures of Wang, Young, and Panjer (1997), the Haezendonck risk measure of Goovaerts and Vyncke (2004).

[22] See Brachinger and Weber (1997) for a survey.

References

Acerbi, C., 2004, Coherent representations of subjective risk-aversion, in *Risk Measures for the 21st Century* . chap. 10, pp. 147–207 (John Wiley and Sons).

———, and D. Tasche, 2002, On the coherence of expected shortfall, *Journal of Banking and Finance* 26, 1487–1503.

Artzner, Philippe, Freddy Delbaen, Jean-Marc Eber, and David Heath, 1997, Thinking coherently, *RISK* 10, 68–71.

———, 1999, Coherent measures of risk, *Mathematical Finance* 9, 203–228.

Baumol, William J., 1963, An expected gain-confidence limit criterion for portfolio selection, *Management Science* 10, 174–182.

Bawa, V.S., 1975, Optimal rules for ordering uncertain prospects, *Journal of Financial Economics* 2, 95–121.

Brachinger, H. W., and M. Weber, 1997, Risk as a primitive: a survey of measures of perceived risk, *Operations Research Spectrum* 19, 235–251.

Carr, P., H. Geman, and D.B. Madan, 2001, Pricing and hedging in incomplete markets, *Journal of Financial Economics* 62, 131–168.

Delbaen, Freddy, November 1998, Coherent risk measures on general probability spaces, Working Paper, ETHZ.

Domar, Evsey D., and Richar A. Musgrave, 1944, Proportional income taxation and risk-taking, *Quaterly Journal of Economics* 58.

Fama, Eugène, 1965, Portfolio analysis in a stable paretian market, *Management Science* 11, 404–419.

Fishburn, P.C., 1977, Mean-risk analysis with risk associated with below target returns, *American Economic Review* 67, 116–126.

Frittelli, Marco, and Emanuela Rosazza-Gianin, 2002, Putting order in risk measures, *Journal of Banking and Finance* 26, 1473–1486.

Gaivoronski, Alexei A., and Gerog Pflug, 2000, Value-at-risk in portfolio optimization: Properties and computational approach, .

Giacometti, R., and S. Ortobelli-Lozza, 2004, *Risk Measures for Asset Allocation Models* (John Wiley and Sons).

Goovaerts, M.J., R. Kaas, and J. Dhaene, 2002, Economic capital allocation derived from risk measures, Leuven University.

Goovaerts, Marc, and David Vyncke, 2004, Actuarial risk measures, Catholic University of Leuven.

Heath, David, 2000, Back to the future, Plenary lecture at the the First World Congress of the Bachelier Society, Paris.

Holton, GLyn A., 2003, *Value-at-Risk: Theory and Practice* (Academic Press).

Jean, William H., 1975, A general class of three-parameter risk measures: Comment, *Journal of Finance* 30, 224–226.

Jia, J., and J. S. Dyer, 1996, A standard measure of risk and risk-value models, *Management Science* 42, 1691–1705.

Jorion, P., 2000, *Value at Risk: The New Benchmark for Managing Financial Risk* (Mac Graw Hill).

Knight, F. H., 1921, *Risk, Uncertainty, and Profit* (Boston).

Markowitz, H., 1952, Portfolio selection, *Journal of Finance* pp. 77–91.

———, 1959, *Portfolio Selection: Efficient Diversification of Investments* (Wiley).

Ortobelli, Sergio, Isabella Huber, and Eduardo Schwartz, 2002, Portfolio selection with stable distributed returns, *Mathematical Methods of Operations Research* 55, 265–300.

Penza, P., and V.K. Bensal, 2001, *Measuring Market Risk with Value-at-Risk* (John Wiley and Sons) financial engineering edn.

Roy, A. D., 1952, Safety first and the holding of assets, *Econometrica* 20.

Stone, B.K., 1973, A general class of three-parameter risk measures, *Journal of Finance* 28, 675–685.

Wang, S. S., V. R. Young, and H. H. Panjer, 1997, Axiomatic characterization of insurance prices, *Insurance, Mathematics and Economics* 21, 173–183.

Yitzhaki, Shlomo, 2002, The gmd: A superior measure of variability for non-normal distributions, Hebrew University.